Height Pairings

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1 Introduction

Notation

- K = number field with integers $\mathcal{O}$, places v, completions $K_v \supset \mathcal{O}_v$, residue field k_v of size q_v,
- $\eta = \mathrm{Spec}(\overline{K})$,
- X = smooth, projective variety over K of dimension n,
- $X_\eta = X \times_K \overline{K}$, $X_v = X \times_K K_v$,
- $Z^i(X)$ = cycles of codimension i, defined over K
- $\mathrm{CH}^i(X) = Z^i(X)/(\text{rational equivalence})$
- $H^i = H^i_{ét}$
- Let $\mathrm{CH}^i(X)^0 = \ker(cl\colon \mathrm{CH}^i(X) \to H^i_{et}(X_\eta, \mathbb{Z}_l(i)))$, (to be safe, let's have l not lie under a place of bad reduction) (TODO: What is known about indep. of l?)
- $Z^i(X)^0$ = preimage of $\mathrm{CH}^i(X)^0$ in $Z^i(X)$.

Goal: We want to construct a pairing

$$\langle \cdot, \cdot \rangle \colon \mathrm{CH}^i(X)^0 \times \mathrm{CH}^{n-i+1}(X)^0 \to \mathbb{R},$$

generalizing the Neron-Tate pairing on abelian varieties.

Note that our cycles are of a dimension where their expected intersection has dimension -1.

Example 1.1 ([9], [3]) *Let C/K be a smooth projective curve, with $\infty \in C(K)$ giving $i\colon C \hookrightarrow \mathrm{Pic}^0(C)$. Let $\langle \cdot, \cdot \rangle_{NT}\colon \mathrm{Pic}^0(C)(K) \times \mathrm{Pic}^0(C)(K) \to \mathbb{R}$ be the Neron-Tate height pairing, identifying $\widehat{\mathrm{Pic}^0(A)} \cong \mathrm{Pic}^0(A)$ via the theta divisor. Then, once we have defined $\langle \cdot, \cdot \rangle$, we will have*

$$\langle P - \infty, Q - \infty \rangle = \langle i(P), i(Q) \rangle_{NT}.$$

Example 1.2 *More generally, for an abelian variety A/K of dimension n, the maps $\mathrm{CH}^1(A)^0 \cong \widehat{A}(K)$, $\mathrm{CH}^n(A)^0 \to A(K)$ identify the pairings $\langle \cdot, \cdot \rangle$ with $\langle \cdot, \cdot \rangle_{NT}\colon A(K) \times \widehat{A}(K) \to \mathbb{R}$.*

Recall that Neron-Tate pairings came from the canonical height function associated to the Poincare bundle. The Poincare bundle is not ample, although it is when restricted to the "diagonal" of $A \times \widehat{A}$ via a polarization $A \to \widehat{A}$. In particular, we get an induced pairing $A(K) \times A(K) \to \mathbb{R}$ which is symmetric and non-degenerate modulo torsion.

To achieve this goal, Beilinson constructs partial local pairings. In other words, for representative cycles C_1, C_2 with disjoint support, we want to define, for every place v, a number $\langle C_1, C_2\rangle_v \in \mathbb{R}$. Then we may define the global pairing as $\langle C_1, C_2\rangle = \sum_v \langle C_1, C_2\rangle_v$.

We would need this formula to descend to Chow.

n=1 For divisors on curves, we could insist that $\langle D, div(f)\rangle = \log|f(D)|$, the sum of the value of f at the points of D, with sign and multiplicity. If this were true for every local pairing v, this would descend to Pic^0 by the product formula.

n $>$ 1 More generally, we could hope that our local pairings were well-behaved with respect to correspondences in $X \times \mathbb{P}^1$, to reduce the descent to Chow to dimension 1.

We can do this as follows:

- For v a place of good reduction (there exists a smooth model over $\mathcal{O}_v$), we may extend cycles C_1, C_2 to a good model by Zariski closure, obtaining $\tilde{C_1}, \tilde{C_2}$. Then define the local pairing in terms of the intersection pairing:

$$\langle C_1, C_2\rangle_v = -(\tilde{C_1} \cdot \tilde{C_2}) \log q_v.$$

- For v a place of bad reduction, we might attempt similarly to choose a regular proper model, extend by Zariski closure. This would be incorrect, even for curves.

  Better: extend to cycle whose intersection with the fiber is cohomologous to zero. (TODO: what does this really mean?) Unfortunately, this is not known to be possible. Conjectural solutions!

- For v an infinite place, there is a construction in terms of Hodge theory. We will present this construction for curves.

Conjecture 1.3 (Beilinson [1]) *For each place v, and smooth projective variety X over K, there exists a local pairing*

$$\langle \cdot, \cdot\rangle_v \colon Z^i(X_v)^0 \times Z^{n-i+1}(X_v)^0 \dashrightarrow \mathbb{R},$$

defined on cycles of disjoint support, such that

1. *if X has good reduction at v, it is the pairing above,*

2. *it is functorial in correspondences $f \subset X \times Y$, in that $\langle f_* a, b\rangle_v = \langle a, f^* b\rangle_v$.*

Lemma 1.4 ([1]) *If local pairings valued in $\mathbb{R}$ exist, satisfying 1.3, then there exists a unique pairing*

$$\langle \cdot, \cdot\rangle \colon \mathrm{CH}^i(X)^0 \times \mathrm{CH}^{n-i+1}(X)^0 \to \mathbb{R},$$

defined, on representative cycles with disjoint support, to be $\sum \langle \cdot, \cdot\rangle_v$.

Remark 1.5 *This is a frustrating definition: we would like it to be positive-definite, but to compute $\langle x, x\rangle$ requires a knowledge of moving! Naive heights for points on varieties don't require moving! At least for divisors alg. equiv. to zero and zero-cycles, Neron has related it to naive heights on Picard/Albanese varieties, similarly to 1.1.*

Remark 1.6 *We needed to have a projective variety so that we can move things: roughly, one can move ample things since they are hyperplane sections. See [11, Tag 0B0D], for a precise statement of Chow's Moving Lemma (TODO: Does this moving lemma work for us?).*

Without assuming other conjectures, we can use one of the following groups instead to define local pairings unconditionally:

1. $\ker(cl\colon Z^i(X_v) \to H^i(X_{K_v^{ur}}, \mathbb{Z}_l(i)))$ (cycles vanishing in *absolute* etale cohomology over the maximal unramified extension of K_v)

2. Image of the fiberwise cohom. to zero cycles in $\mathrm{CH}^i(X_{\mathcal{O}_v})$ (cycles which extend to a local regular model) inside of CH.

The construction under group 1) is not known to lie in $\mathbb{Q} \cdot \log q_v$, only in $\mathbb{Q} \cdot \log q_v$, nor to be independent of l. Despite this lack of practical value, we will sketch it later.

When are these groups equal to $\mathrm{CH}^i(X)^0$?

1. when X is a curve, $CH^i(X)^0$ = group 1 = group 2)

2. when X satisfies the weight-monodromy conjectures at all primes of bad reduction, for the relevant cohomology groups, $CH^i(X)^0$ = group 1.

Warning 1.7 *There is a natural map $\mathrm{CH}^n(X)^0 \to Alb(X)(K)$, but it is not known to be an isomorphism. However, Beilinson shows that, for $X/\mathbb{Q}$, $\mathrm{CH}^n(X)^0 \otimes \mathbb{Q} \cong Alb(X)(\mathbb{Q}) \otimes \mathbb{Q}$ is implied by his conjectures. Even over $\mathbb{C}$, the Albanese is merely the universal abelian variety quotient of $CH^n(X)$.*

Properties

- Correspondences: $f \subset X \times Y$, $\langle f_* a, b\rangle = \langle a, f^* b\rangle$

- Pairing for cycles algebraically equivalent to zero can be reduced to Neron-Tate pairings

Of course, the point of all this is to formulate

Conjecture 1.8 (Beilinson [1])

1. *(Swinnerton-Dyer) The groups $CH^i(X)^0$ are finitely generated, with rank $= ord_{s=0} L(H^{2i-1}(X)(i), s)$.*

2. *The height pairing on $\mathrm{CH} \otimes \mathbb{R}$ is non-degenerate.*

3. *Its determinant, times the determinant of the (real) period matrix for $H^{2i-1}(X)$, equals the leading coefficient of the L-function $L(H^{2i-1}(X)(i), s)$ at $s = 0$.*

But we won't talk more about this.

Question 1.9 *How does Beilinson decide to consider all cycles homologous, not just algebraically equivalent to, zero?*

2 Local Pairings on Curves

2.1 Neron Local Heights [3]

Notation

- X = smooth, projective curve over a local field K_v.
- $\mathrm{Div}(X/K_v)$ = divisors on X rational over K_v,
- $Z^0(X/K_v)$ = free abelian group on $X(K_v)$,
- $|\cdot|_v$ the normalized valuation,
- for f is a function on X, with divisor $div(f)$ relatively prime to $a = \sum m_x(x) \in Z^0(X/K_v)$, we define $f(a) = \prod f(x)^{m_x}$.

Remark 2.1 *We may assume our curve has a rational point, by taking a finite field extension. We will see how the pairing then descends back to K_v.*

Proposition 2.2 (Neron [3]) *There is a unique function $\langle a, b\rangle_v$ on relatively prime divisors $a \in Z^0(X/K_v)$, $b \in \mathrm{Div}^0(X/K_v)$ with values in $\mathbb{R}$ satisfying:*

1. *(Linearity) $\langle a, b\rangle_v + \langle a, c\rangle_v = \langle a, b + c\rangle_v$.*
2. *(Symmetry) $\langle a, b\rangle_v = \langle b, a\rangle_v$ whenever $b \in Z^0(X/K_v)$*
3. *(Principal Divisors) $\langle a, div(f)\rangle_v = \log |f(a)|_v$*
4. *(Continuity) Fix b and a point $x_0 \in X(K_v) - |b|$. Then the function $X(K_v) - |b| \to \mathbb{R}$, defined by $x \mapsto \langle (x) - (x_0), b\rangle_v$, is continuous.*

The proof of uniqueness is easy - the difference of two such functions gives, by fixing the left variable, a continuous homomorphism from $J(K_v)$ to $\mathbb{R}$, which must be constant (no compact subgroups of $\mathbb{R}$).

An easy lemma, using uniqueness:

Lemma 2.3 *For a finite extension H_v/K_v, we have*

$$\langle \cdot, \cdot\rangle_{H_v} = [H_v : K_v]\langle \cdot, \cdot\rangle_{K_v}.$$

After defining this partial pairing on $Z^0(X/K_v) \times \mathrm{Div}^0(X/K_V)$, we may extend it to a partial pairing on $\mathrm{Div}^0(X/K_v) \times \mathrm{Div}^0(X/K_v)$ by using the previous lemma.

2.2 Finite Places

Let K be a p-adic field, X a smooth proper curve over K. We consider a (proper) regular model X_O of X, with special fiber X_k. We will still sometimes denote the place as v.

Definition 2.4 *Let Z_1, Z_2 be effective divisors on X_O, considered as closed subschemes defined by ideal sheaves I_1, I_2. When Z_1 and Z_2 are relatively prime, the interesection $Z_1 \cap Z_2$ is dimension zero, necessarily supported in the special fiber X_k. This is equivalent to $I_1 \cap I_2$ being a k-module of finite length. In this case, we define*

$$Z_1 \cdot Z_2 = \mathrm{len}_k(I_1 \cap I_2).$$

This pairing is in fact biadditive, and extends bilinearly from effective divisors to all divisors.

Definition 2.5 *We say that an effective divisor is* vertical *if its underlying reduced subscheme is contained in X_k. Let $V(X_{\mathcal{O}}) \subset \mathrm{Div}(X_{\mathcal{O}})$ denote the subgroup generated by effective vertical divisors.*

We then have an exact sequence:

$$0 \to V(X_{\mathcal{O}}) \to \mathrm{Div}(X_{\mathcal{O}}) \to \mathrm{Div}(X) \to 0.$$

If we restrict this partial pairing

$$\mathrm{Div}(X_{\mathcal{O}}) \times \mathrm{Div}(X_{\mathcal{O}}) \dashrightarrow \mathbb{Z}$$

to $V(X_{\mathcal{O}}) \times \mathrm{Div}(X_{\mathcal{O}})$, it turns out that it extends to a full pairing

$$V(X_{\mathcal{O}}) \times \mathrm{Div}(X_{\mathcal{O}}) \to \mathbb{Z}.$$

We will not define this pairing, but the following result lists some of its properties.

Proposition 2.6 ([10]) *Let E, F be divisors on X_O with E vertical. Then one has:*

1. *if F is a vertical divisor then $E \cdot F = F \cdot E$,*
2. *if E is prime (subscheme is reduced and irreducible) then $E \cdot F = \deg(\mathcal{O}(F) \otimes \mathcal{O}_E)$,*
3. *if F is principal then $E \cdot F = 0$*

Now, consider the special fiber X_k. Its irreducible components $E_1, \ldots, E_r$ need not be reduced or geometrically connected as k-schemes. We write E_i for the corresponding divisor, which is a multiple of the divisor of the underlying reduced subscheme E_i^{red}. Then $\mathbb{Z}$-linear combinations of E_i^{red} generate the vertical divisors $V(X_{\mathcal{O}})$.

When we restrict the intersection pairing to vertical divisors, we have the following properties:

Theorem 2.7 ([10])

1. *$X_k \cdot F = 0$ for all vertical divisors F,*
2. *$E_i \cdot E_j \geq 0$ if $i \neq j$ and $E_i^2 < 0$,*
3. *the bilinear form given by the intersection product of $Div(X/K) \otimes_{\mathbb{Z}} \mathbb{R}$ is negative semi-definite, with isotropic cone equal to the line generated by X_k.*

Lemma 2.8 *A degree zero divisor $D \in \mathrm{Div}^0(X/K)$ extends uniquely to an element $\tilde{D}$ of $Div(X_O) \otimes \mathbb{Q}$ such that $\tilde{D} \cdot E_i = 0$ for all irreducible components E_i of the special fiber.*

Proof.

$$0 \to \ker(\Sigma) \to \oplus_i \mathbb{Q} \cdot E_i \xrightarrow{\Sigma} \mathbb{Q} \to 0$$

The intersection form is nondegenerate when restricted to $\ker(\Sigma)$.

Let D' the extension of D by Zariski closure, and consider $v = (D' \cdot E_1, \ldots, D' \cdot E_r)$

We have that

$$\Sigma(v) = \sum_i D' \cdot E_i = D' \cdot X_k.$$

Now, $D' \cdot X_k$ may be identified with the degree of the line bundle $\mathcal{O}(D')$ restricted to X_k. Since $\mathcal{O}(D')$ is a locally free sheaf on a proper flat family, this is the same as the degree of $\mathcal{O}(D')_K = \mathcal{O}(D)$, which is zero by assumption.

Thus there is a divisor C supported only on the irreducible components with the same intersection, and we take $D' = D - C$. □

Definition 2.9 *Given D_1, $D_2 \in \mathrm{Div}^0(X_K)$, we define the local pairing*

$$\langle D_1, D_2 \rangle_v = -(D_1 \cdot D_2) \log q_v \in \mathbb{Q} \cdot \log q_v,$$

where $q_v = \#(k)$.

Linearity Obvious

Principal Divisors In case of good reduction:

Let $D_1 = div(f), D_2 = \sum P_i - \sum Q_j \in Z^0(X/K)$. Normalize f so that $|f|_\pi = 1$ (valuation along special fiber), so that $\overline{f} = f \mod \pi$ is a rational function on X_{k_v}, not identically zero.

Then $-(D_1 \cdot D_2) \log q_v = \sum_{P_i \cap X_{k_v}} -ord_{P_i}(\overline{f}) \log q_v - \sum_{Q_j \cap X_{k_v}} -ord_{Q_j}(\overline{f}) \log q_v = \log |f(D_2)|_v$.

In bad reduction, note that f may have zeros or poles on entire irreducible components. These are precisely the ones we need to add/subtract to D_1 for D_1 to intersect each component in degree zero. This is why we cannot naively extend divisors by Zariski-closure, in the case of bad reduction! The rest should be similar...

Continuity We check in the case $D = div(f)$. Fix $x_0 \in X(K) - |D|$. We must show that $x \mapsto \langle (x) - (x_0), D \rangle$ is a continuous function on $X(K) - |D|$. By the previous property, we have $\langle (x) - (x_0), D \rangle = \log |f(x)|_v - \log |f(x_0)|_v$. Since $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous and $|\cdot|_v : K \to \mathbb{R}_{>0}$ is tautologically continuous, it remains to see that $f \colon X(K) - |D| \to K$ is continuous, and this is passage to K-points on the morphism $f : X - |D| \to \mathbf{A}^1_K$, so it remains to check that passage to K-points carries K-morphisms to continuous maps. But by the method of topologizing the K-points (via gluing on Zariski-open affines), this problem is of local nature and hence reduces to the affine case where it is clear (as polynomials are continuous).

Symmetry Obvious

Remark 2.10 *More generally, for non-principal divisors, we may extend a section of a line bundle once we choose a way to extend the line bundle (choosing a model is a choice of extension of O_X), and interpret the pairing as $log|s(D_2)|$. Extending the line bundle is roughly the same as putting a p-adic metric on it. The condition of degree zero may be thought of as a way to get a "canonical metric", as in the archimedean setting.*

2.3 Infinite Places

2.3.1 Green's Functions via thinly veiled Hodge Theory

It will suffice to construct the local pairing when $K_v = \mathbb{C}$, using $[\mathbb{C} : \mathbb{R}]\langle a, b\rangle_{\mathbb{R}} = \langle a, b\rangle_{\mathbb{C}}$ to define the pairing over $\mathbb{R}$.

We will define this local pairing for all Riemann surfaces X. A reference for this construction, in more generality, is in [4].

Claim 2.11 *The real integral*

$$\mathrm{Re}\int : H^0(X, \Omega^1) \times H_1(X, \mathbb{R}) \to \mathbb{R}$$

is a perfect pairing.

Proof. As vector spaces over $\mathbb{R}$, $H^0(X, \Omega^1)$ and $H_1(X, \mathbb{R})$ have the same dimension ($2g$). We know that $H_1(X, \mathbb{Z}) \subset \mathrm{Hom}(H^0(X, \Omega^1), \mathbb{C})$ is a lattice: the Jacobian is compact. So $\int : H_1(X, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{C}}(H^0(X, \Omega^1), \mathbb{C})$. Then use that the map $\mathrm{Re} : \mathrm{Hom}_{\mathbb{C}}(H^0(X, \Omega^1), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(H^0(X, \Omega^1), \mathbb{R})$ is an isomorphism. □

Proposition 2.12 *Let $D \in \mathrm{Div}^0(X)$. There exists a unique $\omega_D \in H^0(X - |D|, \Omega^1)$ such that*

1. *ω_D has log poles along D*
2. $\mathrm{Res}(\omega_D) = D$.
3. $\mathrm{Re}\int \omega_D : H_1(X - |D|, \mathbb{Z}) \to \mathbb{R}$ *vanishes.*

Remark 2.13 *For $D = P - Q$, this can be rephrased as a splitting of the residue sequence*

$$0 \to H^1(X, \mathbb{R}(1)) \to H^1(X - |D|, \mathbb{R}(1)) \stackrel{\mathrm{Res}}{\to} (\mathbb{R} \oplus \mathbb{R})/\mathbb{R} \to 0$$

in the category of mixed hodge structures over $\mathbb{R}$.

Proof. We first check uniqueness. Let ω_D, ω'_D be two forms satisfying 1)-3). The difference $\omega_D - \omega'_D$ lies in $H^0(X, \Omega^1)$ by 1) and 2), and has $\mathrm{Re}\int(\omega_D - \omega'_D) = 0$ by 3). But the claim then implies $\omega_D - \omega'_D = 0$.

It will suffice to consider the divisor $D = P - Q$, since all degree zero divisors are sums of these.

We use Riemann-Roch:

$$h^0(K + (P + Q)) - h^0(-(P + Q)) = deg(P + Q + K) - (g - 1) = 2 + (g - 1) = g + 1$$

We see that $H^0(X - |D|, \Omega^1(log))/H^0(X, \Omega^1)$ is 1-dimensional. Take any $\omega \in H^0(X - |D|, \Omega^1(log))$ with $\mathrm{Res}(\omega) = D$.

The map $\mathrm{Re}\int \omega_1 : H_1(X - |D|, \mathbb{Z}) \to \mathbb{R}$ vanishes on the small loops around $|D|$ ($\int dz/z = 2i\pi$), hence factors through $H_1(X, \mathbb{Z})$.

As a functional $H_1(X, \mathbb{Z}) \to \mathbb{R}$, the claim identifies $\mathrm{Re}\int \omega$ with $\mathrm{Re}\int \eta$ for $\eta \in H^0(X, \Omega^1)$. We can take $\omega_D = \omega - \eta$. Now, $\mathrm{Re}\int \omega_D : : H_1(X - |D|, \mathbb{Z}) \to \mathbb{R}$ vanishes. □

Now, the vanishing of $\mathrm{Re}\int \omega_D : H_1(X - |D|, \mathbb{Z}) \to \mathbb{R}$ implies that as a function on X, $x \mapsto \mathrm{Re}\int_{x_0}^{x} \omega_D$ is well-defined up to a constant. In other words, we may evaluate it on degree-zero divisors.

Definition 2.14 *Given two degree-zero divisors D_1, D_2 on X, we define*

$$\langle D_1, D_2\rangle_\infty = 2 \cdot \mathrm{Re} \int_{D_2} \omega_{D_1}.$$

Let's check that this is a Neron local height:
Linearity: Obvious
Continuity: Obvious
Principal Divisors: $D = div(f), \omega_D = d\log(f), 2 \cdot \mathrm{Re} \int \omega_D = \log|f|^2$
Symmetry:

Proposition 2.15 (Green's Theorem [2][4]) *Let a_i, b_i, $i = 1, \dots, g$, be a symplectic basis for X, thought of as the boundary of a standard fundamental domain. Let ω_1, ω_2 be differential forms satisfying 1) and 2) of 2.12 (no condition on the periods).*

Then integrals $\int_{D_i} \omega_j$ are well-defined mod $\mathbb{Z}(1)$, and we have the following formula:

$$\int_{D_1} \omega_2 - \int_{D_2} \omega_1 = \frac{1}{2\pi i} \sum_{i=1}^{g} \left(\int_{a_i} \omega_1 \cdot \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \cdot \int_{b_i} \omega_1 \right) \mod \mathbb{Z}(1)$$

In particular, if ω_1, ω_2 have purely imaginary periods, then

$$\int_{D_1} \omega_2 = \int_{D_2} \omega_1 \mod \mathbb{R}(1)$$

Proof. Our proof will only prove $mod\mathbb{R}(1)$, and this is all that is needed for our purposes.

First, two lemmas:

Lemma 2.16 ([2]) *For any C^∞ differential forms ω_1, ω_2 on X:*

$$\int_X \omega_1 \wedge \omega_2 = \sum_{i=1}^{g} \left(\int_{a_i} \omega_1 \cdot \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \cdot \int_{b_i} \omega_1 \right)$$

We won't show this: it is straightforward to prove.

Lemma 2.17 *For forms ω_1, ω_2 satisfying 1) and 2) of 2.12,*

$$\int_{D_1} \omega_2 - \int_{D_2} \omega_1 = \frac{1}{2\pi i} \int_{X - |D_1| \cup |D_2|} \omega_1 \wedge \omega_2 \mod \mathbb{Z}(1)$$

Assuming these lemmas, we may modify ω_1, ω_2 so that they are C^∞ across $|D_1| \cup |D_2|$, and apply 2.16.

Proof of 2.17 Define $\mathrm{Re}(\omega) = \frac{1}{2}(\omega + \overline{\omega})$, $\mathrm{Im}(\omega) = \frac{1}{2}(\omega - \overline{\omega})$. Observe that $\int_\gamma \overline{\omega} = \overline{\int_\gamma \omega}$ (take holomorphic/antiholomorphic primitives locally, and then it is obvious).

Let $g_{D_i} = \mathrm{Re} \int \omega_i = \int \mathrm{Re}(\omega_i)$

$$2\pi i(g_{D_1}(D_2) - g_{D_2}(D_1)) = \int_{X-|D_1|\cup|D_2|} d(g_{D_1}\omega_2 - g_{D_2}\omega_1)(\text{Residue}) \tag{1}$$

$$= \int_{X-|D_1|\cup|D_2|} \operatorname{Re}(\omega_1)\wedge\omega_2 + \omega_1\wedge\operatorname{Re}(\omega_2) \tag{2}$$

$$\operatorname{Im}\int_{X-|D_1|\cup|D_2|} \operatorname{Re}(\omega_1)\wedge\omega_2 + \omega_1\wedge\operatorname{Re}(\omega_2) = \int_{X-|D_1|\cup|D_2|} \operatorname{Re}(\omega_1)\wedge\operatorname{Im}(\omega_2) + \operatorname{Im}(\omega_1)\wedge\operatorname{Re}(\omega_2) \tag{3}$$

$$= \int_{X-|D_1|\cup|D_2|} \operatorname{Im}(\omega_1\wedge\omega_2) \tag{4}$$

$$g_{D_1}(D_2) - g_{D_2}(D_1) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{X-|D_1|\cup|D_2|} \omega_1\wedge\omega_2\right) \tag{5}$$

$$\tag{6}$$

□

□

Remark 2.18 *It is not hard, using the symmetry of the local pairing, to prove that Weil reciprocity holds after taking absolute values: in other words, if f, g have disjoint divisors, then $|f(div(g))| = |g(div(f))|$.*

There is an approach by Deligne to local heights which takes a pair of metrized line bundles L, M on X (for example, degree zero line bundles have canonical Hermitian metrics, roughly by constructing Green's functions as above) and produces a hermitian vector space $\langle L, M\rangle$. However, this approach requires the full Weil reciprocity as input.

2.3.2 Examples

If we can write down real-valued harmonic functions $|f|$ on X with $div(|f|) = D$, then $\log(|f|) - \operatorname{Re}\int\omega_D$ will be a constant, by the maximum principle for harmonic functions.

1. $\mathbb{P}^1$: $\log|z|$ for $D = (0) - (\infty)$.

2. $\mathbb{C}/\Lambda$: A natural multi-valued function with a pole of order 1 at ∞ is the Weierstrass σ function:

   $$\log\sigma = \int\zeta(z)dz = \int(\int -\wp(z)dz)dz.$$

   We want to modify this so that it is single-valued, i.e. we want to modify σ so that it transforms under Λ by norm 1 elements.

   Our new function (the Klein function) is

   $$k(z) = \Delta(\Lambda)^{1/12}e^{-z\eta(z)/2}\sigma(z),$$

   where $\eta(z)$ is the $\mathbb{R}$-linear extension of $\eta\colon\Lambda\to\mathbb{C}$ by $\eta(\lambda) = \zeta(z+\lambda) - \zeta(z)$.

   Claim 2.19 $-\log|k(z)| = \frac{1}{2}\operatorname{Re}(z\eta(z)) - \log|\sigma(z)| - \frac{1}{12}\log(\Delta(\Lambda))$ *will be a real-analytic function on E, $\sim\log|z|$ near $z = 0$.*

   When we have a degree zero divisor D, we can use translates of this function to compute $\langle D,\cdot\rangle_{\mathbb{C}}$. For example, if $D = \sum m_i z_i$, then $\operatorname{Re}\int\omega_D = \log|\prod_i k(z-z_i)^{m_i}|$.

3 Beilinson Heights

Pragmatic Motivation: We cannot do intersection pairing unconditionally in the case of bad reduction. We introduce one possible abstract pairing which, when it works, does not require choosing a model, and agrees with intersection pairings when those work too. Unfortunately, it is not known to be independent of l (but that hasn't stopped us before!)

Theological Motivation 1 (Unconditional Archimedean Heights) The same definitions produce the local pairing at infinite places, when applied to Deligne cohomology (extensions of mixed hodge structures) instead of absolute etale cohomology (extensions of l-adic representations). See [1].

Theological Motivation 2 (Weight-Monodromy implies unconditional p-adic Heights) Very similar definitions, when applied to p-adic etale cohomology ($p = l$), produce local heights valued in $\mathbb{Q}_p$, which give a cohomological interpretation of various "p-adic Green's functions", and allow for the statement of p-adic BSD conjectures. See [6].

3.1 Etale Abel-Jacobi Maps

Goal: Algebraic cycles create extension of Galois representations.

Let X be a smooth, proper variety over a field K, with η the geometric generic point. Let G_K be the absolute galois group of K. Consider the cycle class map

$$cl\colon \mathrm{CH}^i(X) \to H^{2i}(X_\eta, \mathbb{Z}_l(i)).$$

Let $\mathrm{CH}^i_Z(X)$ denote the cycles supported on a fixed codimension i subvariety Z. Then we have also

$$cl\colon \mathrm{CH}^i_Z(X) \to H^{2i}_Z(X_\eta, \mathbb{Z}_l(i)).$$

TODO: reference for etale cycle class maps for singular subvarieties of smooth varieties?

The image is the local cohomology with support on Z. This fits into a long exact sequence

$$\ldots \to H^{2i-1}(X_\eta, \mathbb{Z}_l(i)) \to H^{2i-1}((X-Z)_\eta, \mathbb{Z}_l(i)) \to H^{2i}_Z(X_\eta, \mathbb{Z}_l(i)) \to H^{2i}(X_\eta, \mathbb{Z}_l(i)) \to \ldots,$$

which is compatible with the above maps cl in the obvious way.

In particular, for a cycle W supported on Z which is cohomologous to zero, we obtain by pullback

$$\ldots \to H^{2i-1}(X_\eta, \mathbb{Z}_l(i)) \to E \to \mathbb{Z}_l \to 0.$$

This defines a map

$$\mathrm{CH}^i(X)^0 \to \mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, H^{2i-1}(X_\eta, \mathbb{Z}_l(i)))$$

Similarly, we obtain

$$\mathrm{CH}^{n-i+1}(X)^0 \to \mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, H^{2n-2i+1}(X_\eta, \mathbb{Z}_l(n-i+1))).$$

Now, write $V = H^{2i-1}(X_\eta, \mathbb{Z}_l(i))$, $W = H^{2n-2i+1}(X_\eta, \mathbb{Z}_l(n-i+1))$.

Poincare duality for etale cohomology tells us that $V^* = H^{2n-2i+1}(X_\eta, \mathbb{Z}_l(n-i)) = W(-1)$. In other words, our two maps become

$$j_i\colon\ \mathrm{CH}^i(X)^0 \to \mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, V),$$
$$j_{n-i+1}\colon\ \mathrm{CH}^{n-i+1}(X)^0 \to \mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, V^*(1)),$$

Example 3.1 *When X is a curve, we have*

$$0 \to H^1(X_\eta, \mathbb{Z}_l(1)) \to H^1((X-U)_\eta, \mathbb{Z}_l(1)) \to Div^0_Z(X) \to 0.$$

This is an example of a Kummer map: fixing a basepoint ∞, we obtain a map

$$\kappa : X(K) \to \mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, H^1(X_\eta, \mathbb{Z}_l(1))),$$

which agrees with the classical Abel-Jacobi map followed by the Kummer map for the Jacobian of X (also using poincare duality/duality for Jacobians, i.e. $V = V^*(1)$).

Now, let us specialize to K a p-adic field of residue characteristic $p \neq l$.

Proposition 3.2 ([8]) *If X has potentially good reduction or, more generally, if the purity conjecture for the monodromy filtration on V holds, then* $\mathrm{Ext}^1_{G_K}(\mathbb{Q}_l, V) = 0$.

The proof uses the purity from the Weil conjectures + Tate's Euler characteristic formula.

A wishful digression on mixed extensions

Let K a global field. Pretend we had an category like G_K-reps in which V lived, which did Hodge theory when restricted to the decomp. group at infinity. We will consider $H^1(V) \times H^1(V^*(1)) \to H^2(\mathbb{Q}_l(1))$. If we impose a self-dual Selmer condition (say, f), we would get $H^1(V)_f \times H^1(V^*(1))_f \to 0$ (since $Ext^2(\mathbb{Q}_l, \mathbb{Q}_l(1))$ should inject into product of local versions). We suppose that we have algebraic cycles whose cycle-classes are crystalline, giving extensions $E_1 \in H^1(V)_f, E_2 \in H^1(V^*(1))_f$.

We would then attempt to find a canonical reason for this class to vanish, in terms of a canonical element of $H^1(\mathbb{Q}_l(1))$. We could try to do this locally at each place v.

The fact the a cup-product of extensions vanishes implies that we can fill in the upper-right of the following matrix, to get a cochain valued in 3×3 matrices (different coordinates have different coefficients):

$$\begin{pmatrix} 1 & \mathbb{Z}_l(1) & * \\ 0 & 1 & \mathbb{Z}_l \\ 0 & 0 & 1 \end{pmatrix},$$

where the upper-left minor is the extension $\mathrm{Ext}^1_{G_K}(V, \mathbb{Z}_l(1))$ and the lower-right is $\mathrm{Ext}^1_{G_K}(\mathbb{Z}_l, V)$. More diagramatically, but with less meaning, we could write

$$\begin{pmatrix} \mathbb{Z}_l(1) & E_1 & E_3 \\ 0 & V & E_2 \\ 0 & 0 & \mathbb{Z}_l \end{pmatrix},$$

with E_1, E_2 denoting the 1-extensions as above (we have dualized E_2 here). This means only that there is a Galois module E_3 which has an injection $E_1 \hookrightarrow E_3$, a surjection $E_3 \twoheadrightarrow E_2$, but has only 1 copy of V in it. Such an object is called a "mixed extension". There is a canonical mixed extension associated to $E_1 \cup E_2$ when E_1 and E_2 come from cycle classes, but we won't need this.

If $H^0(K_v, V) = H^1(K_v, V)_f = 0$ for every v (for example, we could assume 3.2), we would be in business: Take the "mixed extension" E_3. These assumptions let us turn the restriction of E_3 to G_{K_v} into an element of $H^1(K_v, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l$ (except when $v \mid l$, where we ignore this issue and switch to a different l). This is because our extensions E_1 and E_2 are trivialized as G_{K_v}-modules, even canonically so. This lets us put E_3 into a canonical shape: $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

again diagramatically as
$\begin{pmatrix} \mathbb{Z}_l(1) & 0 & E_3 \\ 0 & V & 0 \\ 0 & 0 & \mathbb{Z}_l \end{pmatrix}$.

At this point, E_3 has become an element of $Ext^1_{G_{K_v}}(\mathbb{Z}_l, \mathbb{Z}_l(1)) \cong \widehat{\mathbb{Q}_l} \cong \mathbb{Z}_l$, by Kummer theory and the ord_l map. This is a local height (after normalizing by $\log q_v$).

Then just add up these numbers: this is a global height.

3.2 Linking Numbers

Now, changing notation, let K be the maximal unramified extension of a p-adic field.

We note that the etale cohomology of $\mathcal{O}_K$ is quite like that of a disk:

Claim 3.3

1. $\mathrm{Spec}(K)$ *has* $\mathbb{Q}_l$ *cohomological dimension 1.*
2. $H^1(\mathrm{Spec}(K), \mathbb{Q}_l(1)) \cong \hat{K} \otimes \mathbb{Q}_l \cong \mathbb{Q}_l$
3. $H^1(\mathrm{Spec}(K), \mathbb{Q}_l(1)) \cong H^2_s(\mathrm{Spec}(\mathcal{O}_K), \mathbb{Q}_l(1))$, *for s the closed point*

Proof. We consider the residue sequence

$$H^1(\mathrm{Spec}(O_K), \mathbb{Q}_l(1)) \to H^1(\mathrm{Spec}(K), \mathbb{Q}_l(1)) \to H^2_s(\mathrm{Spec}(\mathcal{O}_K), \mathbb{Q}_l(1)) \quad (7)$$
$$\to H^2(\mathrm{Spec}(O_K), \mathbb{Q}_l(1)) \to H^2(\mathrm{Spec}(K), \mathbb{Q}_l(1)). \quad (8)$$

1): pro-l inertia is a quotient of $\widehat{\mathbb{Z}}$, of cohomological dimension 1, and for l-torsion modules M, $H^1(\text{pro-p}, M) = 0$ by $l \neq p$.

2): follows from Kummer theory and that $l \neq p$.

For 3): $H^1(\mathrm{Spec}(\mathcal{O}_K), \mathbb{Q}_l(1))$ vanishes, since $\mathrm{Spec}(\mathcal{O}_K)$ is strictly henselian.

$H^2(\mathrm{Spec}(\mathcal{O}_K), \mathbb{Q}_l(1))$ vanishes, since $Br(\mathcal{O}_K) = Br(k) = 0$ (Brauer group of Henselian ring = that of its residue field). □

Let $a_1 \in \mathrm{CH}^i(X)^0$, $a_2 \in \mathrm{CH}^{n-i-1}(X)^0$. Unfortunately, we must make an assumption.

Assumption 3.4 *Assume that a_1, a_2 are zero under the "absolute" cycle class map*

$$cl\colon \mathrm{CH}^i(X)^0 \to H^{2i}(X, \mathbb{Q}_l(i)).$$

Claim 3.5 *Assumption 3.4 holds under the conditions of 3.2.*

Proof. Pretend that K = p-adic field, as opposed to maximal unramified extension. The proof becomes a little more tedious otherwise, but the result is still true, using that our cycles were defined over the p-adic field anyways, and that $\mathrm{Spec}(\mathbb{F}_q)$ has cohomological dimension 1.

The cycle class maps should have the following compatibility:

$$\begin{array}{ccc} \mathrm{CH}^i(X) & \xrightarrow{cl} & H^{2i}(X, \mathbb{Q}_l(1)) \\ & \searrow^{cl} & \downarrow \\ & & H^{2i}(X_\eta, \mathbb{Q}_l(1))^{G_K} \end{array}$$

The vanishing $\mathrm{Ext}^1_{G_K}(\mathbb{Q}_l, V) = 0$ implies that $H^{2i}(X, \mathbb{Q}_l(1)) = H^{2i}(X_\eta, \mathbb{Q}_l(i))^{G_K}$, by Leray-Serre spectral sequence and 1) of 3.3.

Therefore, since our classes were cohomologous to zero in $H^{2i}(X_\eta, \mathbb{Q}_l(1))$, they are also zero in $H^{2i}(X, \mathbb{Q}_l(1))$. □

Claim 3.6 *Assumption 3.4 holds when a_1, a_2 extend to cycles homologous to zero on a regular model $X_\mathcal{O}$.*

Proof.

$$\begin{array}{ccc} \mathrm{CH}^i(X_\mathcal{O}) & \xrightarrow{cl} & H^{2i}(X_\mathcal{O}, \mathbb{Q}_l(1)) \\ \downarrow & & \downarrow \\ \mathrm{CH}^i(X) & \xrightarrow{cl} & H^{2i}(X, \mathbb{Q}_l(1)) \end{array}$$

□

Claim 3.7 *A cycle is homologous to zero on a regular model if and only if its intersection with X_k is homologous to zero on X_k.*

Sketch The natural restriction map $H^{2i}(X_\mathcal{O}) \to H^{2i}(X_k)$ is given by cup-product with the fundamental class of X_k in $X_\mathcal{O}$, hence agrees with intersection on cycle classes in $H^{2i}(X_\mathcal{O})$. But the map $H^{2i}(X_\mathcal{O}) \to H^{2i}(X_k)$ is an isomorphism, by proper base change. □

From the analogs of the long exact sequences above, we have that $cl(a_1) \in H^{2i}_{|a_i|}(X, \mathbb{Q}_l(i))$ is the image of some $\alpha_1 \in H^{2i-1}(X - |a_1|, \mathbb{Q}_l(i))$. Similarly, $cl(a_2)$ is the image of some α_2.

Definition 3.8 *The local linking number $\langle a_1, a_2\rangle_v$ is defined as follows:*

We have $\alpha_1 \cup cl(a_2) \in H^{2n+1}_{|a_2|}(X - |a_1|, \mathbb{Q}_l(n+1))$. The linking number is its image under

$$H^{2n+1}_{|a_2|}(X - |a_1|, \mathbb{Q}_l(n+1)) \to H^{2n+1}(X, \mathbb{Q}_l(n+1)) \stackrel{Tr}{\to} H^1(\mathrm{Spec}(K), \mathbb{Q}_l(1)) \cong \mathbb{Q}_l \cdot \log q_v,$$

where the first map is via excision and the last is by the identification above. Note that we normalize by the size q_v of the residue field of the local field we originally cared about.

For excision in etale cohomology, see [5]. TODO: Find a good reference for the trace/Poincare duality in absolute etale cohomology. Less canonically, can use trace map plus being the only component of a Leray-Serre spectral sequence...

Remark 3.9 *We could have phrased this via "mixed extensions", which would remain in the language of extensions of galois representations as in the previous section, at the price of being confusing. It would involve the Galois structure of $H^{2n}_{|a_2|}((X - |a_1|)_\eta, \mathbb{Q}_l(n+1))$ being standardized by the trivializations of 3.2.*

3.3 Linking = Intersection

See [7] (2.16) for more details, especially pertaining to sign conventions.

The following diagram commutes, where the maps δ come from LES of relative cohomology, the upward maps are restriction, and when coefficients are not written they should be $\mathbb{Q}_l(i)$ or $\mathbb{Q}_l(n-1+1)$:

$$
\begin{array}{cccccc}
H^{2i}_{|a_1|}(X) & \times H^{2n-2i+1}(X-|a_2|) & \xrightarrow{\cup} & H^{2n+1}_{|a_1|}(X-|a_2|) & \xrightarrow{Tr} & H^1(\mathrm{Spec}(K),\mathbb{Q}_l(1)) \\
\uparrow & \uparrow & & \uparrow & & \Big\downarrow \delta \\
H^{2i}_{|a_1|}(X_{\mathcal{O}}) & \times H^{2n-2i+1}(X_{\mathcal{O}}-|a_2|) & \xrightarrow{\cup} & H^{2n+1}_{|a_1|}(X_{\mathcal{O}}-|a_2|) & & \\
\downarrow \cong & \downarrow \delta & & \downarrow \delta & & \\
H^{2i}_{|a_1|}(X_{\mathcal{O}}) & \times H^{2n-2i+2}_{|a_2|}(X_{\mathcal{O}}) & \xrightarrow{\cup} & H^{2n+2}_{|X_k|}(X_{\mathcal{O}}) & \xrightarrow{Tr} & H^2_{|s|}(\mathrm{Spec}(\mathcal{O}),\mathbb{Q}_l(1))
\end{array}
$$

Then, if we start with classes (α, β) in the middle row, mapping down and to the right recovers the intersection product, and mapping up and to the right recovers the linking number (before it is normalized by $\log q_v$).

Thus we see that the linking number and the intersection number agree when one of our cycles a extends to a cycle $\tilde{a}$ on $X_{\mathcal{O}}$ which is cohomologous to zero, so that its cycle class $cl(\tilde{a}) \in H^{2n-2i+2}_{|\tilde{a}|}(X_{\mathcal{O}})$ is in the image of some $\beta \in H^{2n-2i+1}(X_{\mathcal{O}} - |\tilde{a}|)$.

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