Motives and Deligne's Conjecture, Part 2: Examples

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Last week we stated Deligne's conjecture, which says that (under some hypotheses) a special value of the L-function of a motive is a rational multiple of the determinant of a period matrix. In these notes we will work out what this means concretely in two special cases: Artin L-functions and modular forms.

1 Artin Motives and Artin L-Functions

For motivation we present the following identity.

$$1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \frac{1}{10^3} - \frac{1}{11^3} + \cdots = \sum_k \left(\frac{1}{(3k+1)^3} - \frac{1}{(3k+2)^3} \right) = \frac{4\pi^3\sqrt{3}}{243}.$$

We will see that this illustrates a case of Deligne's conjecture.

We'll start with the following simple task: describe Artin motives ("motives of dimension 0") over a number field K. We will see that a reasonable category of dimension-0 motives is equivalent to a category of finite-dimensional Galois representations. We will be interested in the determinant of the period matrix of such objects.

We construct the category of Artin motives over K as follows. Consider first the category $\mathcal{C}$ whose objects are schemes X, smooth of dimension 0 over K. Any such X is a finite disjoint union of spectra of fields finite over K. Given two such objects X and Y one may consider correspondences from X to Y, defined as closed subschemes of the product $X \times Y$. Since $X \times Y$ is a finite disjoint union of points the study of correspondences is purely combinatorial; in particular we do not need to quotient out by numerical equivalence or any such equivalence, and the entire theory (such as it is) will be developed without appeal to unproven conjectures.

In any case, one forms the rational correspondence group in the obvious way, and takes this rational correspondence group to be $\mathrm{Hom}(X, Y)$ in $\mathcal{C}$. Then one forms the category $\mathcal{M}$ of Artin motives from this $\mathbb{C}$ by formally introducing elements (X, p) for every idempotent $p \in \mathrm{Hom}(X, X)$.

We aim to prove the following result.

Lemma 1.1. *The category of Artin motives over a number field K, with coefficients in another number field E, is equivalent to the category of representations of the absolute Galois group of K on E-vector spaces.*

First let's review Galois theory. Recall that a *finite étale K-algebra* is a K-algebra which is a finite direct sum of fields, each finite over K.

Theorem 1.2. *There is an equivalence of categories between the category of finite sets with a continuous action of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, and the category of finite étale K-algebras. Specifically, from a finite étale K-algebra L we obtain the finite set $P(L) = \mathrm{Hom}(L, \overline{\mathbb{Q}})$ with the Galois structure induced from $\overline{\mathbb{Q}}$; in the other direction, given a Galois set I, we obtain the algebra*

$$A(I) = \left(\overline{\mathbb{Q}}^I\right)^G$$

of G-invariant $\overline{\mathbb{Q}}$-valued functions on A, where G acts on both A and $\overline{\mathbb{Q}}$.

Before we pass to motives, let's consider the periods of these zero-dimensional varieties. We restrict ourselves to the situation where L is a field, finite over $\mathbb{Q}$, which we take as our base field. The calculation is completely trivial.

The algebraic de Rham cohomology of $\operatorname{Spec} L$ is

$$H^0_{DR}(\operatorname{Spec} L) = H^0(\operatorname{Spec} L, \mathcal{O}_{\operatorname{Spec} L}) = L.$$

The Betti cohomology is computed after base change to $\mathbb{C}$; the base change of $\operatorname{Spec} L$ to $\mathbb{C}$ is the spectrum of

$$L \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus \mathbb{C}^\sigma,$$

where the direct sum is over all $d = [L : \mathbb{Q}]$ embeddings σ of L into $\mathbb{C}$, and the map from L to the factor $\mathbb{C}^\sigma$ is exactly σ. This spectrum has d points, which give a basis for the rational Betti H_0, which in turn is dual to the Betti cohomology H^0.

The period map can be expressed as a pairing between de Rham H^0 and Betti H_0, given by evaluation. Explicitly, de Rham H^0 is equal to L, which can be viewed as a vector space over $\mathbb{Q}$ with some basis $e_1, \ldots, e_d$. The de Rham class e_i, after base change to $\mathbb{C}$ and evaluation on the point σ_j, gives the complex number $\sigma_j(e_i)$; and the matrix of these numbers is the period matrix. The determinant of this matrix is, up to rational multiple, the square root of the discriminant. However this motive does not satisfy the criticality hypothesis so there is no corresponding L-function value in this case.

At this point we should remark on the functorial relation between de Rham and Betti cohomology. To do this we rephrase the Betti cohomology as follows: the (rational) Betti cohomology is the set of $\mathbb{Q}$-valued functions on a finite disjoint union of points indexed by the maps $\sigma : L \to \mathbb{C}$. If $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ then we can without harm replace $\mathbb{C}$ with $\overline{\mathbb{Q}}$, so the Betti cohomology is exactly

$$\mathbb{Q}^{\mathrm{Hom}(L, \overline{\mathbb{Q}})}.$$

The de Rham cohomology can be recovered from the Betti cohomology via the relation

$$H_{DR} = \left(H_B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}\right)^G,$$

where G, the absolute Galois group of K, acts on both

$$H_B = \mathbb{Q}^{\mathrm{Hom}(L, \overline{\mathbb{Q}})} = \mathbb{Q}^{P(L)}$$

and $\mathbb{Q}$.

This relation will be useful when we compute the de Rham cohomology of motives.

1.1 Correspondences and Representations

To construct the category of Artin motives we need to consider correspondences on $X_1 \times X_2$, where each X_i is the spectrum of an étale K-algebra L_i. Here $X_1 \times X_2$ is the spectrum of $L_1 \otimes_K L_2$, which is again an étale K-algebra. So $X_1 \times X_2$ is a finite disjoint union of points; a closed subscheme is exactly a subset of those points, so a rational cycle is a $\mathbb{Q}$-valued function on the set of those points. By base change to the complex numbers, we see that the group of rational cycles on $X_1 \times X_2$ can be viewed as the Galois-invariant elements

$$\left(\mathbb{Q}^{\mathrm{Hom}(L_1 \otimes_K L_2, \overline{\mathbb{Q}})}\right)^G.$$

By Galois theory, this is equal to

$$\left(\mathbb{Q}^{P(L_1 \otimes_K L_2)}\right)^G = \left(\mathbb{Q}^{P(L_1)} \otimes \mathbb{Q}^{P(L_2)}\right)^G = \mathrm{Hom}_G(\mathbb{Q}^{P(L_2)}, \mathbb{Q}^{P(L_1)}).$$

The functor from Artin motives to Galois representations will be given by the Betti cohomology. As we have seen, a field L corresponding to a finite Galois set $P(L)$ has Betti cohomology the corresponding permutation representation $\mathbb{Q}^{P(L)}$. Hence, the space of correspondences is equal to

$$\mathrm{Hom}_G(H_B(\mathrm{Spec}\, L_2), H_B(\mathrm{Spec}\, L_2)).$$

For the construction of Artin motives with coefficients in an arbitrary number field E, one also needs to know that the algebra of cycles with coefficients in E is

$$\left(\mathbb{E}^{\mathrm{Hom}(L_1 \otimes_K L_2, \overline{\mathbb{Q}})}\right)^G = \mathrm{Hom}_G(\mathbb{E}^{P(L_2)}, \mathbb{E}^{P(L_1)}) = \mathrm{Hom}_G(H_B(\mathrm{Spec}\, L_2), H_B(\mathrm{Spec}\, L_2)).$$

Now the category of Artin motives with coefficients in E is constructed from the category $\mathcal{C}$ as the category of pairs (X, p) with X an object of $\mathcal{C}$ and P an idempotent in its endomorphism algebra. This $\mathcal{C}$ has for objects exactly the permutation representations of the Galois group G – in other words, the representations of G on E^S, for S a finite set with a G-action. And the endomorphisms of these objects are exactly their usual endomorphisms in the sense of representation theory. Now, for any representation X and subrepresentation $V \subseteq X$, there is a G-equivariant projection p_V of X onto V. Hence, this pair gives rise to a motive (X, p). We identify (X, p) with the representation V. One sees immediately that V is indeed the Betti cohomology of the motive (X, p).

Again, some compatibilities must be verified: we must check that for any pairs (X, p) and (Y, q), the morphisms between them in $\mathcal{M}$ are naturally identified with the G-equivariant homomorphisms between the corresponding vector spaces. This is straightforward and we omit it.

Lemma 1.1 now follows from the fact that every continuous E-linear representation of the group G is contained in a permutation representation.

1.2 Periods and Deligne's Conjecture

Now we can return to Deligne's conjecture. We will only state it for Artin motives of rank 1, that is to say, for one-dimensional Galois representations. So, let ϵ be a character of the absolute Galois group of K.

First we need to discuss criticality. Criticality is a hypothesis to Deligne's conjecture; it is a condition on the Hodge structure of the motive in question. For the details, see last week's notes.

To state Deligne's conjecture we must suppose that our motive is defined over $\mathbb{Q}$; otherwise we must take a Weil restriction down to $\mathbb{Q}$, as explained last week. We will limit ourselves to the case of Artin representations of $\mathbb{Q}$.

The Hodge structure of Artin motives is very simple: all the cohomology occurs in H^{00}. If X is an Artin motive then the Tate twist $X(a)$ has all its cohomology in H^{aa}. Recall the criticality condition in this context. Let F_∞ denote the action on Betti cohomology of complex conjugation on our motive. The motive is critical if and only if F_∞ acts as ϵ on the Betti cohomology, where ϵ is taken to be -1 if $a \geq 0$ and $+1$ otherwise.

In terms of Galois representations, one may check that F_∞ corresponds to complex conjugation. (Recall that we were treating $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ for the purposes of computing Betti cohomology, so complex conjugation specifies a well-defined involution of $\overline{\mathbb{Q}}$.)

Hence, we obtain the following. If $F_\infty = -1$ on X, then $X(a)$ is critical if a is a nonnegative even integer or a negative odd integer (e.g. -3, -1, 0, 2). If $F_\infty = +1$ on X, then $X(a)$ is critical if a is nonnegative and odd, or negative and even (e.g. -4, -2, 1, 3). Deligne's conjecture predicts the values of Artin L-functions at these values (up to rational multiple).

Our next task is to compute the periods. We use the functorial relationship

$$H_{DR}(X) = \left(H_B(X) \otimes_E \overline{\mathbb{Q}}\right)^G.$$

To justify this one must verify the following functoriality: any correpsondence induces maps on both Betti and de Rham cohomology; the map on de Rham cohomology is exactly the map induced on de Rham cohomology from the map on Betti cohomology by the relationship above. Once this has been verified it follows that the relationship holds for arbitrary Artin motives.

So, consider our character ϵ of the Galois group of $\mathbb{Q}$, and let X be the corresponding motive. Let $V_{\mathbb{Q}}$ denote a one-dimensional $\mathbb{Q}$-vector space on which G acts by ϵ, and let V denote the base change of $V_{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$, using the Galois action on $\overline{\mathbb{Q}}$. Explicitly, let e span $V_{\mathbb{Q}}$. Then for any $a \in \overline{\mathbb{Q}}$ and $\sigma \in G$, the Galois action is given by

$$\sigma(ae) = \epsilon(\sigma)\sigma(a)e.$$

To compute the de Rham cohomology we need to find a Galois-fixed element of V. A natural approach is to start with an arbitrary element of V and take the average of its Galois conjugates. Of course if we start with e (and ϵ is nontrivial) we will get 0 as this average; we need to find some other element of V which gives a nonzero average.

Remark 1.3. *Alternatively, we could have approached this de Rham cohomology by the following method: First, we would compute explictly the effect of correspondences on de Rham cohomology. Suppose first we are given a correspondence from L_1 to L_2 that is given explictly as a direct factor of $L_1 \otimes L_2$ – note that our idempotent correspondences will in general be rational linear combinations of correspondences of this form. Then the pull-push form on de Rham says that the map on cohomology, from L_2 to L_1, is given by the following rule: embed L_2 into the tensor product, project down to the direct factor, and then trace down to L_1. For arbitrary correspondences one takes the appropriate linear combination of projections. To carry out the calculation in this way it is necessary to translate our Galois character into the language of Artin motives as étale algebras with idempotent correspondences. We introduced the functorial relationship between Betti and de Rham cohomology exactly to avoid this unpleasant task.*

But in this alternative approach, the problem of finding a basis for the de Rham cohomology of our motive translates to finding a basis for the image, one-dimensional over $\mathbb{Q}$, of our idempotent *map induced on de Rham $L \to L$. This map factors as a trace from $L \otimes L$ to L, so it's enough to find some element of L that after this trace operation gives a nonzero image in L. This is essentially the situation we find ourselves in.*

What element of V (a torsor for $\overline{\mathbb{Q}}$) should we try? Inspired by the remark above, note that ϵ splits over $L = \mathbb{Q}(\zeta_f)$, where f is the conductor of ϵ and ζ_f a primitive f-th root of unity. So maybe it's not unreasonable to try

$$v' = \zeta_f e.$$

Averaging over G gives the following sum

$$v = \left(\sum_{u \in (\mathbb{Z}/f\mathbb{Z})^*} \epsilon(u) \zeta_f^u \right) e,$$

where we view ϵ as a Dirichlet character by means of the usual identification

$$\mathrm{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q}) \equiv (\mathbb{Z}/f\mathbb{Z})^*.$$

The sum above is a Gauss sum, and it is known to have nonzero value. Hence the period in question, up to rational multiple, is

$$d(X) = \sum_{u \in (\mathbb{Z}/f\mathbb{Z})^*} \epsilon(u) \zeta_f^u.$$

In this case, Deligne predicts that the Artin L-function will have

$$L(X, s) \in \mathbb{Q}^* (2\pi)^s i^\epsilon d(X)$$

for all critical values of s. (Here ϵ is either 0 or 1; an explicit formula is given in [Del].)

1.3 Our Example

Recall the identity

$$1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \frac{1}{10^3} - \frac{1}{11^3} + \cdots = \sum_k \left(\frac{1}{(3k+1)^3} - \frac{1}{(3k+2)^3} \right) = \frac{4\pi^3\sqrt{3}}{243}.$$

The left-hand side is

$$L(X, 3),$$

where X is the quadratic Dirichlet character of discriminant -3. The relevant Galois character ϵ splits over $\mathbb{Q}[\zeta_3]$, so the Gauss sum appearing in Deligne's conjecture has value

$$\zeta_3 - \zeta_3^2 = i\sqrt{3}.$$

This is consistent with the actual value

$$\frac{4\pi^3\sqrt{3}}{243}.$$

2 Modular Forms

Now we consider the case of modular forms. Again, Deligne's conjecture in this case will turn out to be known. We will show that there is a motive associated to any modular form of weight 2. (The result is true, but more difficult, for modular forms of arbitrary weight.) Deligne's conjecture will follow, assuming a couple of standard results in the theory of modular forms.

Our first goal is to arrive at a proof of the following theorem, in the case $k = 2$.

Theorem 2.1. *Let $f = \sum a_n q^n$ be a holomorphic new cusp eigenform of weight k, and suppose the a_n all lie in the number field E. Then there is a motive $M(f)$ with coefficients in E whose L-function is $\sum a_n n^{-s}$.*

The idea of the proof is as follows. For $k = 2$, modular forms give rise to global differentials on the modular curve M (or its completion $\overline{M}$); in other words, they naturally lie in H^1_{DR}. Modular forms are eigenvectors for the Hecke operators, which are exactly the endomorphisms of H^1 induced by the Hecke correspondences. Since modular forms can be described in terms of kernels and images of elements of the Hecke algebra, there are corresponding motives. By a multiplicity one result, we know that the Hecke eigenvalues cut out exactly the form f (instead of a higher-dimensional space of modular forms). And by Eichler-Shimura, we know that the L-function of the motive agrees with that of the modular form.

For $k > 2$ even, one proceeds as follows. Then modular forms of weight k naturally lie in $H^1(X)$, where X is the $k-1$-fold product of the universal elliptic curve with itself over M. But in this situation the failure of X to be complete presents a more serious technical obstacle: it is necessary to find a smooth projective completion of X to which the Hecke correspondences extend. This can be done in general but it requires some effort and the details will not interest us. See [Sch].

2.1 Modular Curve and Hecke Correspondences

To make matters precise, suppose we have a modular form f of weight $k = 2$ and level N. Let's suppose for concreteness that f is a modular form for $\Gamma_1(N)$.

Let M be the modular curve for $\Gamma_1(N)$. This curve can be expressed as the quotient of the complex upper half-plane by the action of the group $\Gamma_1(N)$. Alternatively, it is a coarse moduli space for elliptic curves with N-level structure. Loosely speaking, an "N-level structure" on an elliptic curve means a cyclic degree-N subgroup of the N-torsion of the curve. For a precise statement, see [KM]. When we need to make the dependence on N explicit we will write $M(N)$ for M.

Let p be a prime not dividing N. We wish to define the Hecke correspondence T_p on $M(N)$. This T_p will be a closed subscheme of $M(N) \times M(N)$. In terms of moduli, consider an R-valued point (E_1, E_2) of $M(N) \times M(N)$, with each E_i an elliptic curve with N-level structure. This point will belong to $T_p(R)$ if (and only if) there is a degree-p isogeny from E_1 to E_2 respecting the level structure on each.

in another language, we can define our correspondence by giving a curve with two maps to $M(N)$. The curve will be $M(pN)$. The first map doesn't change the elliptic curve, and replaces the cyclic pN-element group with its unique N-element subgroup. The second map replaces the elliptic curve with its quotient by the p-element subgroup defined by the level structure.

Like all quasiprojective curves, $M(N)$ has a canonical smooth projective model $\overline{M(N)}$. This $\overline{M(N)}$ has, again, a moduli interpretation, and has been very well studied. The Hecke correspondence extends to the projective curve $\overline{M(N)}$. For details see [DR] or [KM].

The Hecke correspondences T_p, like all correspondences, induce maps on the cohomology of $M(N)$ for every cohomology theory. Hecke eigenforms are defined to be simultaneous eigenvectors for all the T_p (which commute) acting on the space of modular forms. By strong multiplicity one, the simultaneous eigenspaces for the T_p (even omitting those p not dividing N, for which we have not defined the Hecke operators) are one-dimensional. So, any eigenform f spans such an eigenspace.

Since the cohomology of $M(N)$ is finite-dimensional, given any Hecke correspondence T_p with an eigenvalue λ lying in a number field E, one can produce a polynomial in T_p, with coefficients in E, which projects onto the corresponding eigenspace. Hence, given our modular form f with coefficients in E, we can produce a correspondence $C(f)$ on $M(N)$ with coefficients in the same E whose action on cohomology is to project onto the one-dimensional span of f.

Now we have constructed the motive $M(f)$; the Eichler-Shimura relation tells us that the L-functions agree. This proves the theorem.

2.2 Deligne's Conjecture for Modular Forms

Now we need to compute the periods of $M(f)$. Let's suppose for simplicity that f has coefficients in $\mathbb{Q}$.

In de Rham cohomology, f corresponds to something of Hodge type $(1, 0)$; its conjugate (which also lies in $M(f)$) is of type $(0, 1)$. This f is a global differential form on M, which we need to integrate against a rational Betti cycle that is fixed by F_∞.

We will see that such a Betti cycle is given, in the upper half-plane, by the imaginary axis.

First of all, we can see from the moduli interpretation that the complex conjugation action L_∞ on the upper half-plane is given by $z \mapsto -\overline{z}$. So the imaginary axis is indeed fixed by F_∞. The axis has as endpoints the cusps at 0 and $i\infty$. If the modular group happened to identify these cusps (as, for example, does the full modular group $\Gamma(1)$) then the imaginary axis would indeed represent an F_∞-invariant Betti cycle.

In general it does not. What is true, though, is that the difference between the two cusps is of finite index in the Jacobian of the modular curve (Manin-Drinfeld). This means that the integral of any global differential (like f) along this curve is a rational multiple of its integral along a Betti cycle, which is again F_∞-invariant.

Since Deligne's conjecture is only up to rational multiples anyway this presents no issue.

So we have constructed a rational Betti cycle γ on $M(N)$, invariant under F_∞. We need such a cycle on the subspace $M(f)$; up to rational multiple, there is only one such. Now $M(N)$ projects onto $M(f)$, so if our cycle projects to something nonzero it will do the job. As a matter of fact that is good enough: we will show that the critical L-value $L(f, 1)$ is in any case equal to

$$\int_0^{i\infty} f(z)dz.$$

If γ is nonzero in $M(f)$, then the above integral is exactly the determinant of the one-by-one period matrix in question. If γ is zero in $M(f)$ then the above integral is also equal to zero; so $L(f, 1) = 0$ is a rational multiple of the period whatever that period may be.

2.3 The L-function of a Modular Form

First, a formal calculation to suggest why the integral above should give an L-value. We write $f(z) = \sum a_n e^{2\pi i n z}$, where since f is cuspidal the sum ranges over $n > 0$. Then

$$\int_0^{i\infty} f(z)dz = \int \sum a_n e^{2\pi i n z} dz = \sum \int a_n e^{2\pi i n z} dz = \sum \frac{a_n}{2\pi n} = \frac{1}{2\pi i} L_0(f, 1).$$

Here L_0 represents the part of the L-function coming from finite places; as usual the extra 2π is swallowed by a gamma-factor, and Deligne's conjecture is proved.

Of course this is not a proof because of pesky convergence issues. The sum appearing above is divergent in general. We have not even shown that the L-function has the analytic continuation needed to even talk about the value $L(f, 1)$.

We now make the argument precise.

By elementary results on modular forms one knows that the coefficients a_n are bounded by something of the form $n^{k/2}$, and thus the L-function converges on some right half-plane.

Formally, we have

$$\int_0^{\infty} y^s f(iy) \frac{dy}{y} = \sum \int a_n y^s e^{-2\pi n y} \frac{dy}{y} = \sum \frac{a_n}{(2\pi)^s} n^s \Gamma(s) = \Gamma_{\mathbb{C}}(s) L_0(f, s) = L(f, s).$$

Since f is a cusp form, its Fourier expansion has leading term $e^{-2\pi i z}$, which decays exponentially as z approaches $i\infty$. So the integral on the left-hand side above (with the y^s term) converges as y goes to infinity, no matter how large s is.

To deal with the other end of the integral it is desirable (though not strictly necessary) to introduce the Atkin-Lehner involution w (sometimes written w_N to emphasize the dependence on N). In terms of moduli, we want to find $w(E, G)$, where E is an elliptic curve and G a cyclic order-N subgroup. Take for the new elliptic curve $E' = E/G$; the N-torsion of E maps to a cyclic order-N subgroup G' of the N-torsion of E'. We take $w(E, G) = (E', G')$. We see immediately that w is an involution.

In terms of the upper half-plane, w_N interchanges the points z and $\frac{-1}{Nz}$.

Every Hecke eigenform is also an eigenform for w_N. (This may be built into the definition of eigenform or regarded as a consequence of strong multiplicity one.)

As a first consequence, we consider the behavior of the integral above near $y = 0$. Under the involution w the cusp 0 moves to $i\infty$, and the integral becomes

$$\int_0^{\infty} N^{-s} y^{k-s} (wf)(iy) \frac{dy}{y}.$$

Now we have seen that this integral converges as y goes to ∞, which is to say that the first integral converges as y goes to 0, regardless of the value of s.

This implies immediately that our integral formula converges, and hence is valid, for all s for which the L-series converges. Furthermore, the integral converges for all s, so it may be used to define an analytic continuation, valid for all s. This implies Deligne's conjecture.

As a side note, the second formula above, coupled with $wf = \pm f$, proves the functional equation

$$L(k - s) = \pm N^{-s} L(s),$$

where the sign is given by the eigenvalue of w on f.

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