THE EQUIVALENCE OF ARTIN–TATE AND BIRCH–SWINNERTON-DYER CONJECTURES

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Contents

We follow the ideas of Gordon [3] (who seems to follow the sketch of Tate [6, p.427-430]), with minor modifications:

(1) We make a simplifying assumption that the fibered surface $X \to C$ has a section;
(2) We introduce the notion of Néron-Severi structures to clean up the combinatorics.

1. Statement of the conjectures

1.1. **Setup.** Let k be a finite field with q elements. Let $\sigma \in \mathrm{Gal}(\overline{k}/k)$ denote the geometric Frobenius. Let $p = \mathrm{char}(k)$ and let $\ell \neq p$ be a prime number.

Let C be a smooth, projective and geometrically connected curve over k of genus g. Let $F = k(C)$ be the function field of C. Let $|C|$ be the set of closed points of C. For $v \in |C|$, let $\mathcal{O}_v, F_v, k(v)$ be the completed local ring at v, its fraction field and residue field.

Let X be a smooth projective surface over k and $f : X \to C$ be a flat morphism such that the generic fiber X_F is a smooth and geometrically connected curve over F. Assume also:

- $X(F) \neq \varnothing$; equivalently f admits a section $\gamma : C \to X$.

Let $\mathrm{NS}(X)$ be the Néron-Severi group of X, i.e., divisors on X modulo algebraic equivalence. This is a finitely generated abelian group equipped with a non-degenerate symmetric bilinear pairing into $\mathbb{Z}$.

Notation: for any *free* abelian group Λ with a non-degenerate symmetric bilinear pairing $\langle -, - \rangle : \Lambda \times \Lambda \to \mathbb{C}$ we denote

$$\mathrm{Disc}(\Lambda) := |\det(\langle \lambda_i, \lambda_j \rangle)| \tag{1.1}$$

be the absolute value of the Gram matrix formed by any $\mathbb{Z}$-basis $\{\lambda_i\}$ of Λ.

1.2. **Fact** (Raynaud [5, Th 7.2.1])**.** In our situation, f is cohomologically flat, i.e., for any geometric point $s \in C$, we have $h^0(X_s, \mathcal{O}_{X_s}) = 1$ and $h^1(X_s, \mathcal{O}_{X_s}) = n$ is independent of s. Therefore $\mathbf{R}^1 f_* \mathcal{O}_X$ is locally free of rank n over C.

1.3. **Conjecture** (Artin-Tate [6, p. 426, (C)])**.**

(1) *Let* $P_2(X, q^{-s}) := \det(1 - \sigma q^{-s} | \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell))$. *Then*

$$\mathrm{ord}_{s=1} P_2(X, q^{-s}) = \mathrm{rk}\mathrm{NS}(X).$$

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(2) *The Brauer group* $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ *is finite;*
(3) *As* $s \to 1$*, we have*

$$P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\mathrm{rk}\mathrm{NS}(X)} \frac{\mathrm{Disc}(\mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tor}})}{|\mathrm{NS}(X)_{\mathrm{tor}}|^2} \cdot q^{-\alpha(X)} \cdot |\mathrm{Br}(X)|.$$

where $\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim \mathrm{Pic}^{\circ}_{X/k}$.

1.4. **Jacobian and Néron model.** Let $A_F = \mathrm{Pic}^0_{X_F/F}$ be the Jacobian of the curve X_F. This is a principally polarized abelian variety over Spec F. Let A be its Néron model over C. Let A° be the fiberwise neutral component of A. Let $\omega_{A/C}$ be the sheaf of relative top differential forms on $A \to C$, viewed as a line bundle on C via the identity section.

For each place v of C, let A_v denote the fiber of A over Spec $k(v)$. Let $c_v = [A_v(k(v)) : A^{\circ}_v(k(v))]$ be the Tamagawa factor at v.

1.5. **Fact** ([1]). In our situation, the relative Picard functor $\mathrm{Pic}_{X/C}$ is represented by an algebraic space. Let $\mathrm{Pic}^{\circ}_{X/C}$ be the fiberwise neutral component of $\mathrm{Pic}_{X/C}$. Then $\mathrm{Pic}^{\circ}_{X/C}$ is a smooth group scheme of finite type over C. The canonical map $\mathrm{Pic}^{\circ}_{X/C} \to A^{\circ}$ (by the Néron mapping property) is an isomorphism.

1.6. **Conjecture** (Birch–Swinnerton-Dyer [6, p.419, (B)]).
(1) *Let* $L(s, A_F)$ *be the complete* L*-function for the abelian variety* A_F *over* F*. Then*

$$\mathrm{ord}_{s=1} L(s, A_F) = \mathrm{rk} A(F).$$

(2) $\text{Ш}(A_F)$ *is finite.*
(3) *As* $s \to 1$*, we have*

$$L(s, A_F) \sim (s-1)^{\mathrm{rk} A(F)} \frac{\mathrm{Disc}_{NT}(A(F)/A(F)_{\mathrm{tor}})}{|A(F)_{\mathrm{tor}}|^2} \cdot (\prod_v c_v) \cdot q^{-\deg \omega_{A/C} - n(g-1)} \cdot |\text{Ш}(A_F)| \tag{1.2}$$

In (1.2) we put Disc_{NT} to emphasize that we are using the Néron-Tate pairing on $A(F)/A(F)_{\mathrm{tor}}$, which takes values in $\mathbb{Q}\log(q)$ (see §5.3). If we divide the Néron-Tate by $\log(q)$ (we call it the modified NT pairing), we denote the discriminant of the resulting pairing on $A(F)/A(F)_{\mathrm{tor}}$ simply by $\mathrm{Disc}(A(F)/A(F)_{\mathrm{tor}})$. After change of variables $s \mapsto q^{-s}$, (1.2) is equivalent to

$$L(s, A_F) \sim (1 - q^{1-s})^{\mathrm{rk} A(F)} \frac{\mathrm{Disc}(A(F)/A(F)_{\mathrm{tor}})}{|A(F)_{\mathrm{tor}}|^2} \cdot (\prod_v c_v) \cdot q^{-\deg \omega_{A/C} - n(g-1)} \cdot |\text{Ш}(A_F)|. \tag{1.3}$$

We will sketch a proof of the following result.

1.7. **Theorem** (Gordon, conjectured by Tate [6, p.427, (d)]). *Conjectures 1.3 and 1.6 are equivalent.*

2. Néron-Severi structures

2.1. **Definition.** A *Néron-Severi structure* (*NS structure* for short) is a triple (Λ, V, ι) where
(1) Λ is a finitely generated abelian group;
(2) V is a finite-dimensional $\mathbb{Q}_\ell$-vector space with a $\mathbb{Q}_\ell$-linear automorphism σ;
(3) ι is a map of abelian groups $\Lambda \to V^{\sigma}$.

A NS structure (Λ, V, ι) satisfies the *Tate condition* if ι induces an isomorphism

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} V^{\sigma}.$$

There is an obvious notion of morphisms between NS structures, making NS structures an abelian category $\mathcal{NS}$ with a monoidal structure given by the tensor product.

2.2. **Definition.** A *polarization* on a NS structure (Λ, V, ι) is a symmetric bilinear σ-invariant perfect pairing

$$\langle -, - \rangle : V \times V \to \mathbb{Q}_\ell$$

such that its pullback to $\Lambda \times \Lambda$ takes values in $\mathbb{Q}$, and is perfect on $\Lambda_{\mathbb{Q}}$.

Note that a polarization on (Λ, V, ι) forces $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to V^\sigma$ to be injective. Polarized NS structures form an exact category $\mathcal{PNS}$ under the following definition of short exact sequences.

2.3. **Definition.** Let $(\Lambda, V, \iota, \langle -, - \rangle)$ and $(\Lambda_i, V_i, \iota_i, \langle -, - \rangle_i)$ be polarized NS structures. An exact sequence

$$0 \to (\Lambda_1, V_1, \iota_1) \to (\Lambda, V, \iota) \to (\Lambda_2, V_2, \iota_2) \to 0 \tag{2.1}$$

in $\mathcal{NS}$ is an exact sequence in $\mathcal{PNS}$ if the $\langle -, - \rangle$ restricts to $\langle -, - \rangle_1$ on V_1, whose orthogonal complement maps isometrically to $(V_2, \langle -, - \rangle_2)$. In this case we have a canonical orthogonal decomposition $V = V_1 \oplus V_2$ and $\Lambda_{\mathbb{Q}} = \Lambda_{1,\mathbb{Q}} \oplus \Lambda_{2,\mathbb{Q}}$.

2.4. **Lemma.** *If $(\Lambda, V, \iota, \langle -, - \rangle)$ is a polarized NS structure which satisfies the Tate condition, then the generalized σ-eignspace $V^{(\sigma)}$ of V for eigenvalue 1 is equal to V^σ (eigenspace for eigenvalue 1).*

Proof. Replacing V by its generalized eigenspace for eigenvalue 1, the pairing is still perfect there. So we may assume σ is an unipotent element in $\mathrm{SO}(V)$. By Jacobson-Morosov, one can find a homomorphism $\phi : \mathrm{SL}_2 \to \mathrm{SO}(V)$ such that $\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \sigma$. The diagonal torus in SL_2 gives a grading $V = \oplus_i V_i$ such that $\langle V_i, V_j \rangle = 0$ for $i \neq -j$. Clearly $V^\sigma \subset V_{\geq 0}$. If $\sigma \neq 1$, V^σ has nonzero intersection with $V_{>0}$, then $\langle V_{>0} \cap V^\sigma, V^\sigma \rangle = 0$, i.e., the pairing on V^σ is degenerate. This contradicts the non-degeneracy of the pairing on $\Lambda \otimes \mathbb{Q}_\ell = V^\sigma$ (by Tate's condition). □

2.5. **Example.** Let X be a smooth projective surface over a finite field k. Then

$$(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl)$$

is a NS structure where $cl : \mathrm{NS}(X) \to \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^\sigma$ is the cycle class map. The intersection pairing on $\mathrm{NS}(X)$ and the cup product pairing on $\mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))$ with values in $\mathrm{H}^4(X_{\overline{k}}, \mathbb{Q}_\ell(2)) \cong \mathbb{Q}_\ell$ make $(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl, \cup)$ a polarized NS structure. It satisfies the Tate condition if and only if the Tate conjecture holds for divisors on X.

2.6. **Example.** Let $V_{\mathbb{Z}}$ be a finitely generated abelian group with an action of σ. Then $(V_{\mathbb{Z}}^\sigma, V_{\mathbb{Z}} \otimes \mathbb{Q}_\ell, \iota)$ (where ι is the obvious embedding) is a NS structure that satisfies Tate's condition.

2.7. **Definition.** Let $(\Lambda, V, \iota, \langle -, - \rangle)$ be a polarized NS structure. Define

$$\Delta(\Lambda, V, \iota, \langle -, - \rangle) := \det(1 - \sigma | V/V^{(\sigma)}) \cdot \frac{|\Lambda_{\mathrm{tor}}|^2}{\mathrm{Disc}(\Lambda/\Lambda_{\mathrm{tor}}, \langle -, - \rangle)}.$$

Here $V^{(\sigma)}$ is the generalized eigenspace of σ on V for eigenvalue 1. We often abbreviate the above quantity as $\Delta(\Lambda, V)$.

2.8. **More on discriminants.** Why is $\frac{|\Lambda_{\mathrm{tor}}|^2}{\mathrm{Disc}(\Lambda/\Lambda_{\mathrm{tor}})}$ a natural quantity to consider? Recall for a map $f : A \to B$ of abelian groups with finite kernel and cokernel, we can define $z(f) = |\ker(f)|/|\mathrm{coker}(f)|$. For any complex $C_\bullet$ of abelian groups with bounded and finite homology, we may define $z(C_\bullet) = \prod_i |\mathrm{H}_i C|^{(-1)^i}$. Applying this construction to the complex $A \to B$ in degrees 0 and 1 recover $z(A \to B)$. This descends to a homomorphism $z : K_0(D_f(\mathfrak{Ab})) \to \mathbb{Q}^\times$, where $D_f(\mathfrak{Ab})$ is the derived category of complexes in abelian groups with bounded and finite homology.

A $\mathbb{Z}$-valued pairing on Λ gives a map $\Lambda \to \mathbf{R}\mathrm{Hom}(\Lambda, \mathbb{Z})$ whose cone has finite cohomology if the pairing is perfect on $\Lambda \otimes \mathbb{Q}$. It is easy to check that

$$\frac{|\Lambda_{\mathrm{tor}}|^2}{\mathrm{Disc}(\Lambda/\Lambda_{\mathrm{tor}})} = z(\Lambda \to \mathbf{R}\mathrm{Hom}(\Lambda, \mathbb{Z})).$$

This measures how far the pairing $\langle -, - \rangle$ is from being perfect. We define the discriminant of the pairing on Λ to be

$$\mathrm{Disc}(\Lambda) := z(\Lambda \to \mathbf{R}\mathrm{Hom}(\Lambda, \mathbb{Z}))^{-1} = \frac{\mathrm{Disc}(\Lambda/\Lambda_{\mathrm{tor}})}{|\Lambda_{\mathrm{tor}}|^2}.$$

This is consistent with the old definition of Disc in (1.1) for free abelian groups.

If the pairing on Λ is only $\mathbb{Q}$-valued, we may choose n such that the pairing is $\frac{1}{n}\mathbb{Z}$-valued. Then we define

$$\mathrm{Disc}(\Lambda) := \frac{z(\mathbf{R}\mathrm{Hom}(\Lambda,\mathbb{Z}) \to \mathbf{R}\mathrm{Hom}(\Lambda,\frac{1}{n}\mathbb{Z})}{z(\Lambda \to \mathbf{R}\mathrm{Hom}(\Lambda,\frac{1}{n}\mathbb{Z}))}$$

The map on the numerator is given by the inclusion $\mathbb{Z} \hookrightarrow \frac{1}{n}\mathbb{Z}$; the map on the denominator is given by the pairing on Λ. One can check that this is still equal to $\frac{\mathrm{Disc}(\Lambda/\Lambda_{\mathrm{tor}})}{|\Lambda_{\mathrm{tor}}|^2}$ (hence independent of n).

More generally, for a pairing $\Lambda_1 \times \Lambda_2 \to \mathbb{Q}$ between finitely generated abelian groups that is perfect after tensoring with $\mathbb{Q}$, we may define

$$\mathrm{Disc}(\Lambda_1,\Lambda_2) := \frac{z(\mathbf{R}\mathrm{Hom}(\Lambda_2,\mathbb{Z}) \to \mathbf{R}\mathrm{Hom}(\Lambda_2,\frac{1}{n}\mathbb{Z}))}{z(\Lambda_1 \to \mathbf{R}\mathrm{Hom}(\Lambda_2,\frac{1}{n}\mathbb{Z}))}$$

for sufficiently divisible n.

2.9. **Example.** In the situation of Example 2.6, suppose further that $V_{\mathbb{Z}}$ carries a symmetric bilinear $\mathbb{Z}$-valued pairing $\langle -,-\rangle$ that is perfect on $V_{\mathbb{Q}}$. Then we get a polarized NS structure $(V_{\mathbb{Z}}^{\sigma}, V = V_{\mathbb{Q}_\ell}, \iota, \langle -,-\rangle)$. By Lemma 2.4, we have $V^{(\sigma)} = V^{\sigma}$. One easily calculates

$$\det(1-\sigma|V/V^{\sigma}) = \pm z(V_{\mathbb{Z}}^{\sigma} \to V_{\mathbb{Z},\sigma})^{-1}.$$

The sign is positive if σ has finite order. Therefore we have

$$\Delta(V_{\mathbb{Z}}^{\sigma}, V_{\mathbb{Q}_\ell}) = \pm z(V_{\mathbb{Z}}^{\sigma} \to V_{\mathbb{Z},\sigma})^{-1}\mathrm{Disc}(V_{\mathbb{Z}}^{\sigma})^{-1} = \pm \mathrm{Disc}(V_{\mathbb{Z}}^{\sigma}, V_{\mathbb{Z},\sigma})^{-1}.$$

which measures how far the pairing $V_{\mathbb{Z}}^{\sigma} \times V_{\mathbb{Z},\sigma} \to \mathbb{Z}$ (induced from $\langle -,-\rangle$) is from being perfect.

For a smooth projective variety X over k and $V = \mathrm{H}^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i)))$, we may dream for a "motivic cohomology lattice" $V_{\mathbb{Z}} =$ "$\mathrm{H}^{2i}(X_{\overline{k}}, \mathbb{Z}(i))''$ that makes $(V_{\mathbb{Z}}, V)$ into a situation like the one discussed above. Although $V_{\mathbb{Z}}$ does not exist in general, its completion $V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$ exists naturally (as $\mathbb{Z}_\ell$-cohomology if V is $\mathbb{Q}_\ell$-cohomology, for various ℓ), which sometimes works as well as $V_{\mathbb{Z}}$.

The following lemma, which is an easy exercise, describes the behavior of $\Delta(\Lambda, V)$ under "isogeny".

2.10. **Lemma.** *Let $(\Lambda, V, \iota, \langle -,-\rangle)$ be a polarized NS structure. Let $\alpha : \Lambda' \to \Lambda$ be a map of abelian groups with finite kernel and cokernel. Then $(\Lambda', V, \iota\circ\alpha, \langle -,-\rangle)$ is also a polarized NS structure. Moreover, we have*

$$\Delta(\Lambda', V, \iota\circ\alpha, \langle -,-\rangle) = \Delta(\Lambda, V, \iota\circ\alpha, \langle -,-\rangle) z(\Lambda' \to \Lambda)^2.$$

The next two lemmas, although both trivial to prove, are the key properties of $\Delta(\Lambda, V)$ that link the leading term formulae in Conjectures 1.3 and 1.6.

2.11. **Lemma.** *The map $\Delta : \mathrm{Ob}(\mathcal{PNS}) \to \mathbb{Q}_\ell^{\times}$ descends to a homomorphism $\Delta : K_0(\mathcal{PNS}) \to \mathbb{Q}_\ell^{\times}$.*

Proof. We need to check that for an exact sequence (2.1), we have

$$\Delta(\Lambda, V) = \Delta(\Lambda_1, V_1)\Delta(\Lambda_2, V_2). \tag{2.2}$$

Since $\Lambda_{\mathbb{Q}} = \Lambda_{1,\mathbb{Q}} \oplus \Lambda_{2,\mathbb{Q}}$ isometrically, there is a subgroup $\Lambda_2' \subset \Lambda_2$ of finite index such that $\Lambda_1 \oplus \Lambda_2' \subset \Lambda$ isometrically, with cokernel Λ_2/Λ_2'. By Lemma 2.10, we have

$$\frac{\Delta(\Lambda_1 \oplus \Lambda_2', V_1 \oplus V_2)}{\Delta(\Lambda, V)} = [\Lambda_2 : \Lambda_2']^{-2} = \frac{\Delta(\Lambda_2', V_2)}{\Delta(\Lambda_2, V_2)}.$$

Since $\Delta(\Lambda_1 \oplus \Lambda_2', V_1 \oplus V_2) = \Delta(\Lambda_1, V_1)\Delta(\Lambda_2', V_2)$, we conclude (2.2) from the above identity. □

Consider the following situation. Let $(\Lambda, V, \iota, \langle -,-\rangle)$ be a polarized NS structure. Let $(\Gamma, W, \jmath')$ be an isotropic subobject of the NS structure (Λ, V, ι) with the trivial σ-action on W. Let $\Gamma' \subset \Gamma^{\perp} \subset \Lambda$ be a subgroup of finite index, then $(\Gamma', W^{\perp}, \jmath')$ ($\jmath'$ is the restriction of ι to Γ') is a sub NS structure of (Λ, V, ι). The subquotient $(\Gamma'/\Gamma, W^{\perp}/W, \bar{\jmath}, \langle -,-\rangle)$ carries a polarization induced from that of V, hence it is an object of $\mathcal{PNS}$. The following lemma is an easy exercise.

2.12. **Lemma.** *In the above situation, we have*

$$\Delta(\Lambda, V) = \Delta(\Gamma'/\Gamma, W^\perp/W) \cdot \mathrm{Disc}(\Gamma, \Lambda/\Gamma')^{-2}.$$

In particular, if Γ *is free and saturated in* Λ*, and* $\Gamma' = \Gamma^\perp$*, then*

$$\Delta(\Lambda, V) = \Delta(\Gamma^\perp/\Gamma, W^\perp/W).$$

3. Reformulation of the conjectures

Using the notation $\Delta(\Lambda, V)$, we may reformulate Conjecture 1.3 as

3.1. **Conjecture** (Reformulation of Artin-Tate)**.**

(1) *The NS structure* $(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl)$ *satisfies the Tate condition.*
(2) $\mathrm{Br}(X)$ *is finite.*
(3) *We have*

$$\Delta(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl, \cup) = q^{-\alpha(X)}|\mathrm{Br}(X)|.$$

Next we give a reformulation of the Conjecture 1.6 in terms of $\Delta(\Lambda, V)$.

3.2. **Fact** (The F/k-trace, for more details see [2])**.** There is an (unique) abelian variety B over k with a map $B \otimes_k F \to A_F$ and is final for such maps from abelian varieties over k. The above map has finite infinitesimal kernel. We have $B \cong \mathrm{Pic}^\circ_{X/k}/\mathrm{Pic}^0_{C/k}$ (note $\mathrm{Pic}^\circ_{X/k}$ means the neutral component of $\mathrm{Pic}_{X/k}$, for proof see [3, Prop 4.4]). Moreover, if we view $V_\ell(B)$ as a $\mathrm{Gal}(F^s/F)$-module via $\mathrm{Gal}(F^s/F) \to \mathrm{Gal}(\overline{k}/k)$, $V_\ell(B)$ is the maximal submodule of $V_\ell(A)$ (which is semisimple as $\mathrm{Gal}(F^s/F)$-module) on which the action of $\mathrm{Gal}(F^s/F)$ factors through $\mathrm{Gal}(\overline{k}/k)$.

We assume the following statement, which will be proved in §5.

3.3. **Proposition.** *There is a natural map* $\iota : A(F)/B(k) \to \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))^\sigma$ *which makes*

$$(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)), \iota)$$

a NS structure. Moreover, up to a sign, ι *is an isometry with respect to the modified NT pairing on* $A(F)/B(k)$ *and the cup product on* $\mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))$*, the latter giving a polarization* $\langle -, - \rangle_A$ *of the above NS structure.*

Note that $A(F)/B(k)$ is a finitely generated abelian group by the theorem of Lang-Néron.

3.4. **Conjecture** (Refomulation of B-SD)**.**

(1) *The NS structure* $(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)), \iota)$ *satisfies the Tate condition.*
(2) $Ш(A_F)$ *is finite;*
(3) *We have*

$$\Delta(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)), \iota, \langle -, - \rangle_A) = (\prod_v c_v) \cdot q^{-\alpha(A)} \cdot |Ш(A_F)| \tag{3.1}$$

where $\alpha(A) := \deg \omega_{A/C} + n(g-1) + \dim B$.

3.5. **Lemma.** *Conjecture* (3.4) *is equivalent to Conjecture 1.6.*

Proof. The L-function $L(s, A_F)$ is attached to the constructible sheaf $\mathbf{R}^1 f_* \mathbb{Q}_\ell$ over C. By Grothendieck-Lefschetz fixed point formula, we have

$$L(s, A_F) = \frac{\det(1 - \sigma q^{-s} | \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell))}{\det(1 - \sigma q^{-s} | \mathrm{H}^0(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell)) \det(1 - \sigma q^{-s} | \mathrm{H}^2(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell))}.$$

Let $V = \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))$. By Fact 3.2, $V_\ell(B)^*$ as a geometrically constant sheaf on C exhausts the geometrically constant constituents of $\mathbf{R}^1 f_* \mathbb{Q}_\ell$ (as a shifted perverse sheaf on $C_{\overline{k}}$). Therefore $\mathrm{H}^0(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell) \cong V_\ell(B)^*$ as $\mathrm{Gal}(\overline{k}/k)$-modules, and by duality $\mathrm{H}^2(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell) \cong V_\ell(B)^*(-1)$ as $\mathrm{Gal}(\overline{k}/k)$-modules. We get

$$L(s, A_F) = \frac{\det(1 - \sigma q^{1-s} | V)}{\det(1 - \sigma q^{-s} | V_\ell(B)^*) \det(1 - \sigma q^{1-s} | V_\ell(B)^*)}. \tag{3.2}$$

Consider the rank part of the conjectures. Since the eigenvalues of σ on $V_\ell(B)$ are q-Weil numbers of weight -1, the denominator above does not vanish at $s=1$. Therefore $L(s,A_F)$ and $\det(1-\sigma q^{1-s}|V)$ have the same vanishing order at $s=1$, which is $r=\dim V^{(\sigma)}$ (generalized eigenspace). From the fact that $(A(F)/B(k),V,\iota)$ is a NS structure we see that

$$\mathrm{rk}A(F)=\mathrm{rk}A(F)/B(k)\le\dim V^{\sigma}\le\dim V^{(\sigma)}=r \tag{3.3}$$

If the $L(s,A_F)$ has order $\mathrm{rk}A(F)$, then all equalities hold, and in particular $\mathrm{rk}A(F)/B(k)=\dim V^{\sigma}$ which says that $(A(F)/B(k),V,\iota)$ satisfies the Tate condition. Conversely, if $(A(F)/B(k),V,\iota)$ satisfies the Tate condition, then $V^{\sigma}=V^{(\sigma)}$ by Lemma 2.4, hence again all inequalities (3.3) are equalities, and in particular the vanishing order r of $L(s,A_F)$ is $\mathrm{rk}A(F)$.

Now we may assume the rank part of both conjectures and compare the leading coefficients of $L(s,A_F)$ and $\det(1-\sigma q^{1-s}|V)$. By (3.2), we have

$$\lim_{s\to1}\frac{L(s,A_F)}{(1-q^{1-s})^r}=\lim_{s\to1}\frac{\det(1-\sigma q^{1-s}|V)}{(1-q^{1-s})^r}\cdot\frac{1}{\det(1-\sigma q^{-1}|V_\ell(B)^*)\det(1-\sigma|V_\ell(B)^*)}.$$

Since

$$\begin{aligned}\lim_{s\to1}\frac{\det(1-\sigma q^{1-s}|V)}{(1-q^{1-s})^r}&=\det(1-\sigma|V/V^{(\sigma)})=\det(1-\sigma|V/V^{\sigma});\\ \det(1-\sigma|V_\ell(B)^*)&=|B(k)|;\\ \det(1-\sigma q^{-1}|V_\ell(B)^*)&=q^{-\dim B}|B(k)|.\end{aligned}$$

Therefore

$$\lim_{s\to1}\frac{L(s,A_F)}{(1-q^{1-s})^r}=q^{\dim B}|B(k)|^{-2}\det(1-\sigma|V/V^{\sigma}).$$

By definition,

$$\Delta(A(F)/B(k),V)=\frac{\det(1-\sigma|V/V^{(\sigma)})}{\mathrm{Disc}(A(F)/B(k))}=\frac{\det(1-\sigma|V/V^{\sigma})}{\mathrm{Disc}(A(F))|B(k)|^2}.$$

Therefore

$$\begin{aligned}\lim_{s\to1}\frac{L(s,A_F)}{(1-q^{1-s})^r}&=q^{\dim B}\mathrm{Disc}(A(F))\Delta(A(F)/B(k),V)\\&=q^{\dim B}\frac{\mathrm{Disc}(A(F)/A(F)_{\mathrm{tor}})}{|A(F)_{\mathrm{tor}}|^2}\Delta(A(F)/B(k),V)\end{aligned}$$

whence the equivalence of two leading term formulae (1.3) and (3.1). □

4. Structure of the proof

The proof of Theorem 1.7 is accomplished by the following steps.

4.1. **NS structures attached to bad fibers.** Let $v\in|C|$ be a place. Let $X_{v,1},\cdots,X_{v,h_i}$ be the reduced irreducible components, each being an integral curve with field of constants $k(v)_i/k(v)$. Inside $\mathrm{NS}(X)$, write $X_v=\sum_i m_iX_{v,i}$ where m_i is the length of X_v at the generic point of $X_{v,i}$. We may assume that the section $\gamma(C)$ passes (only) through $X_{v,1}$ (at necessarily a $k(v)$ point in the smooth locus of $X_{v,1}$), forcing $k(v)_1=k(v)$ and $m_1=1$.

Let D_v be the free abelian group with basis $\{X_{v,1},\cdots,X_{v,m}\}$. Let

$$\Lambda_v:=D_v/\mathbb{Z}\cdot X_v.$$

Since $m_1=1$, Λ_v is a free-abelian group with a basis given by the images of $\{X_{v,2},\cdots,X_{v,h_i}\}$. Similarly, base changing to $X_v\otimes_k\overline{k}$ (tensor over k, not $k(v)$), we have similar lattices $D_{v,\overline{k}}$ with basis given by the irreducible components of $X_v\otimes_k\overline{k}$ and quotient

$$\Lambda_{v,\overline{k}}:=D_{v,\overline{k}}/\mathrm{Span}\{X_s;s\in C(\overline{k})\text{ over }v\}.$$

Then $\Lambda_{v,\overline{k}}$ is a free abelian group with a σ-action and an intersection pairing. By the Hodge Index Theorem, the intersection pairing on $\Lambda_{v,\overline{k}}$ is negative definite. Clearly $\Lambda_v = \Lambda^{\sigma}_{v,\overline{k}}$. As in Example 2.6, we may form the polarized NS structure

$$(\Lambda_v, \Lambda_{v,\overline{k}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$$

which is zero if X_v is irreducible.

4.2. **Proposition.** *There is a 3-step filtration of the object* $(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl) \in \mathcal{NS}$*:*

$$0 \subset \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset \mathrm{Fil}^0 = (\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl)$$

with the following properties.

- $\mathrm{Fil}^2 = (\mathbb{Z}\Phi, \mathrm{H}^2(C_{\overline{k}}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \cdot cl(\Phi))$*, where* $\Phi \in \mathrm{NS}(X)$ *is the class of any divisor* $f^{-1}(E)$*, where* E *is a divisor on* C *of degree 1;*
- $\mathrm{Fil}^1 = (\mathrm{Fil}^0)^\perp$*,* $\mathrm{Gr}^0 \cong (\mathbb{Z}\gamma(C), \mathbb{Q}_\ell)$*, i.e.,* $\mathrm{Gr}^0\mathrm{NS}(X)$ *is generated by the image of* $\gamma(C)$*;*
- $\mathrm{Gr}^1 = (\mathrm{Fil}^2)^\perp/\mathrm{Fil}^2$ *is a polarized NS structure admitting a short exact sequence in the category* $\mathcal{PNS}$*:*

$$0 \to \oplus_v(\Lambda_v, \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell) \to \mathrm{Gr}^1(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))) \to (A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))) \to 0.$$

Applying Lemma 2.12 to the above filtration and Lemma 2.11 to Gr^1 of the filtration, we get

4.3. **Corollary.** *The NS structure* $(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1)), cl)$ *satisfies the Tate condition if and only if* $(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)), \iota)$ *does. When they satisfy the Tate condition, we have*

$$\Delta(\mathrm{NS}(X), \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))) = \Delta(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))) \prod_v \Delta(\Lambda_v, \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell).$$

Using this corollary, comparing the Conjectures 3.1 and 3.4, their equivalence then follows from the combination of the three statements below.

4.4. **Theorem** (Artin, Grothendieck [4, §4])**.** *There is a canonical isomorphism*

$$Ш(A_F) \cong \mathrm{Br}(X).$$

4.5. **Proposition.** *For each* $v \in |C|$ *we have*

$$\Delta(\Lambda_v, \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell) = c_v^{-1}.$$

4.6. **Proposition.** *We have*

$$\alpha(X) = \alpha(A). \tag{4.1}$$

5. More details

5.1. **Sketch of proof of Prop 3.3 and Prop 4.2.** Leray-spectral sequence for the fibbration $f : X_{\overline{k}} \to C_{\overline{k}}$ degenerates at E_2, giving a filtration on $V := \mathrm{H}^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))$:

$$0 \subset L^2V \subset L^1V \subset L^0V = V$$

with associated graded

$$\mathrm{Gr}^2_L V = \mathrm{H}^2(C_{\overline{k}}, \mathbf{R}^0 f_* \mathbb{Q}_\ell(1)) = \mathrm{H}^2(C_{\overline{k}}, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell;$$
$$\mathrm{Gr}^1_L V = \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)); \mathrm{Gr}^0_L V = \mathrm{H}^0(C_{\overline{k}}, \mathbf{R}^2 f_* \mathbb{Q}_\ell(1)).$$

Note the argument for degeneration at E_2 uses the Hodge Index theorem (negative definiteness of the intersection pairing on $\Lambda_{v,\overline{k}}$).

The sheaf $\mathbf{R}^2 f_* \mathbb{Q}_\ell(1)$ over $C_{\overline{k}}$ admits a further filtration

$$0 \to \oplus_{s \in C(\overline{k})} i_{s,*} W_s \to \mathbf{R}^2 f_* \mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell \to 0.$$

Here each $i_s : \mathrm{Spec}\ \overline{k} \hookrightarrow C_{\overline{k}}$ is the inclusion of a geometric point, and $W_s = \ker(\mathrm{H}^2(X_s, \mathbb{Q}_\ell(1)) \to \mathbb{Q}_\ell)$ given by integration along X_s. We have a natural isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules

$$\oplus_{C(\overline{k})\ni s \text{ over } v} W_s \cong \mathrm{Hom}(\Lambda_{v,\overline{k}}, \mathbb{Q}_\ell)$$

given by the pairing between the divisor group $\Lambda_{v,\overline{k}}$ and the cohomology group $\mathrm{H}^2(X_s, \mathbb{Q}_\ell(1))$, which is perfect by the Hodge Index Theorem. In particular, we have

$$0 \to \oplus_v \mathrm{Hom}(\Lambda_{v,\overline{k}}, \mathbb{Q}_\ell) \to \mathrm{Gr}^2_L V \to \mathbb{Q}_\ell \to 0.$$

If we combine $\mathrm{Gr}^1_L V$ with the $\oplus_{s\in C(\overline{k})} W_s$ part of $\mathrm{Gr}^0_L V$, and renumber the filtration steps, we get a filtration

$$0 \subset \mathrm{Fil}^2 V \subset \mathrm{Fil}^1 V \subset \mathrm{Fil}^0 V = V$$

with $\mathrm{Fil}^2 V = L^2 V$ and $\mathrm{Fil}^1 V = (\mathrm{Fil}^2 V)^\perp$, and

$$0 \to \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)) \to \mathrm{Gr}^1_{\mathrm{Fil}} V \to \oplus_v \mathrm{Hom}(\Lambda_{v,\overline{k}}, \mathbb{Q}_\ell) \to 0. \tag{5.1}$$

We remark that this is the filtration on V induced from the perverse filtration of the complex $\mathbf{R}f_*\mathbb{Q}_\ell$. In general, the perverse filtration behaves better than the filtration $\mathrm{Fil}^i V$ in that it satisfies Poincaré duality and Hard Lefschetz.

We consider a similar filtration on $\mathrm{NS}(X)$:

$$0 \subset \mathrm{Fil}^2\mathrm{NS}(X) \subset \mathrm{Fil}^1\mathrm{NS}(X) \subset \mathrm{Fil}^0\mathrm{NS}(X) = \mathrm{NS}(X)$$

as designated by Prop 4.2, i.e., $\mathrm{Fil}^2\mathrm{NS}(X) = \mathbb{Z}\Phi$ and $\mathrm{Fil}^1\mathrm{NS}(X) = (\mathrm{Fil}^2\mathrm{NS}(X))^\perp$, which consists of all divisor classes with total degree 0 long fibers of f. Since $\gamma(C)\cdot\Phi = 1$, $\mathrm{Gr}^2\mathrm{NS}(X)$ is freely generated by the image of $\gamma(C)$. We have a similar filtration on $\mathrm{NS}(X_{\overline{k}})$.

Let $\mathrm{NS}(X)_{\mathrm{ver}} \subset \mathrm{NS}(X)$ be generated by the irreducible components of all fibers of f. Note that there is a canonical isomorphism.

$$\mathrm{NS}(X)_{\mathrm{ver}}/\mathbb{Z}\Phi \cong \oplus_v \Lambda_v$$

Then we have a further filtration of $\mathrm{Gr}^1\mathrm{NS}(X)$

$$0 \to \oplus_v \Lambda_v \to \mathrm{Gr}^1\mathrm{NS}(X) \to \mathrm{Gr}^1\mathrm{NS}(X)_{\mathrm{hor}} \to 0.$$

where $\mathrm{Gr}^1\mathrm{NS}(X)_{\mathrm{hor}}$ is defined to be the quotient. Similarly, we have an inclusion

$$\oplus_v \Lambda_{v,\overline{k}} \hookrightarrow \mathrm{Gr}^1\mathrm{NS}(X_{\overline{k}}).$$

5.2. **Theorem** (Shioda-Tate, see [3, Prop 4.5], [7, Prop 4.1])**.** *The above map induces an isomorphism*

$$\mathrm{Gr}^1\mathrm{NS}(X)_{\mathrm{hor}} \xrightarrow{\sim} A(F)/B(k).$$

Sketch of proof. Let $\mathrm{Div}_0(X) \subset \mathrm{Div}(X)$ be the divisors whose intersection number with any fiber of f is zero. Restricting a divisor on X to its generic fiber gives $\tau : \mathrm{Div}_0(X) \to \mathrm{Pic}(X) \to \mathrm{Pic}^0(X_F) = A(F)$, which is surjective (set-theoretic inverse given by taking closure). Now let $\mathrm{Div}_{\mathrm{alg}}(X) = \ker(\mathrm{Div}_0(X) \to \mathrm{Fil}^1\mathrm{NS}(X))$ be those divisors algebraically equivalent to zero. Then τ restricts to $\mathrm{Div}_{\mathrm{alg}}(X) \to \mathrm{Pic}^\circ(X) \to A(F)$, the latter map comes from $\mathrm{Pic}^\circ_{X/k} \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ F \to A_F$ hence lands in the F/k-trace B. Therefore $\tau(\mathrm{Div}_{\mathrm{alg}}(X)) \subset B(k)$, and τ induces $\overline{\tau} : \mathrm{Fil}^1\mathrm{NS}(X) \to A(F)/B(k)$. Since vertical divisors are mapped to 0 under τ, τ factors through

$$\mathrm{Gr}^1\mathrm{NS}(X)_{\mathrm{hor}} \to A(F)/B(k).$$

The fact this is an isomorphism follows from the surjectivity of $\mathrm{Pic}^\circ_{X/k} \to B$ (see [3, Prop 4.4]). □

To summarize, we have an exact sequence

$$0 \to \oplus_v \Lambda_v \to \mathrm{Gr}^1\mathrm{NS}(X) \to A(F)/B(k) \to 0. \tag{5.2}$$

Now consider the cycle class map $cl : \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to V$. It is easy to see that this map is strictly compatible with the filtrations denoted by Fil. Now we consider $cl : \mathrm{Gr}^1\mathrm{NS}(X_{\overline{k}}) \to \mathrm{Gr}^1_{\mathrm{Fil}} V$. The map

$$\oplus_v \Lambda_{v,\overline{k}} \subset \mathrm{Gr}^1\mathrm{NS}(X_{\overline{k}}) \xrightarrow{cl} \mathrm{Gr}^1_{\mathrm{Fil}} V \twoheadrightarrow \oplus_v \mathrm{Hom}(\Lambda_{v,\overline{k}}, \mathbb{Q}_\ell)$$

is given by the intersection pairing is an isomorphism after tensoring the source by $\mathbb{Q}_\ell$. Therefore, the exact sequence (5.1) admits a canonical splitting

$$\mathrm{Gr}^1_{\mathrm{Fil}}V \cong (\oplus_v \Lambda_v \otimes \mathbb{Q}_\ell) \oplus \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)). \tag{5.3}$$

Moreover this is an orthogonal decomposition. The exact sequence (5.2) induces

$$\iota : A(F)/B(k) \cong \mathrm{Gr}^1\mathrm{NS}(X)/\oplus_v \Lambda_v \to \mathrm{Gr}^1_{\mathrm{Fil}}V/(\oplus_v \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell) \cong \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))$$

whose image lies in the σ-invariants. This makes $(A(F)/B(k), \mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1)))$ into a NS structure.

5.3. **Fact** (Tate)**.** For $\alpha, \beta \in A(F)$, if we extend them to divisors $\widetilde{\alpha}, \widetilde{\beta}$ (with $\mathbb{Q}$-coefficients in general) with zero intersection with all vertical divisors, then we have

$$\langle \alpha, \beta \rangle_{NT} = -\langle \widetilde{\alpha}, \widetilde{\beta} \rangle \log(q).$$

Since $\widetilde{\alpha}$ is orthogonal to all vertical divisors, $cl(\widetilde{\alpha})$ has image $(0, \iota(\alpha)) \in \mathrm{Gr}^1_{\mathrm{Fil}}V$ under the decomposition (5.3). Therefore, ι respects the negated modified NT pairing on $A(F)/B(k)$ and the cup product pairing on $\mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))$. This finishes the proof of Prop 3.3.

Using the splitting (5.3), we may view $\oplus_v \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell$ as a sub of $\mathrm{Gr}^1_{\mathrm{Fil}}V$ with quotient $\mathrm{H}^1(C_{\overline{k}}, \mathbf{R}^1 f_* \mathbb{Q}_\ell(1))$, forming a new filtration of $\mathrm{Gr}^1_{\mathrm{Fil}}V$ compatible with the filtration (5.2) on $\mathrm{Gr}^1\mathrm{NS}(X)$. This proves Prop 4.2. □

5.4. **Proof of Prop 4.5.** For simplicity let $L = \Lambda_{v,\overline{k}}$. This is the free abelian group with basis given by irreducible components of $X_v \otimes_k \overline{k}$. The σ-action is by permuting this basis. In particular, both L^σ and L_σ are free $\mathbb{Z}$-modules. The NS structure $(\Lambda_v, \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell) = (L^\sigma, L \otimes \mathbb{Q}_\ell)$ fits into Example 2.6. According to the calculations there, we have

$$\Delta(\Lambda_v, \Lambda_{v,\overline{k}} \otimes \mathbb{Q}_\ell) = z(L^\sigma \to \mathrm{Hom}(L_\sigma, \mathbb{Z}))$$

is the reciprocal of the discriminant of the pairing between L^σ and L_σ. The pairing on L gives

$$0 \to L \to L^\vee \to Q \to 0$$

for some finite abelian group Q whose order is $\mathrm{Disc}(L)$. Taking σ-invariants gives an exact sequence

$$0 \to L^\sigma \to (L^\vee)^\sigma \to Q^\sigma \to L_\sigma$$

Since L_σ is free while Q^σ is finite, the last map is zero hence $Q^\sigma = (L^\vee)^\sigma/L^\sigma = \mathrm{Hom}(L_\sigma, \mathbb{Z})/L^\sigma$ whose order is then $z(L^\sigma \to \mathrm{Hom}(L_\sigma, \mathbb{Z}))^{-1}$. We reduce the problem to showing that

$$|Q^\sigma| = c_v = [A_v(k(v)) : A_v^\circ(k(v))] = \pi_0(A_v \otimes_k \overline{k})^\sigma,$$

where the last equality uses Lang's theorem to A_v^0. This follows from the isomorphism of σ-modules ([1, 9.5, Thm 4])

$$Q \cong \pi_0(A_v \otimes_k \overline{k}).$$

Idea of proof: we may replace C by its strict henselization at v, therefore consider X over a strict henselian DVR R with fraction field K and closed point s, and define L accordingly. In this case, the Néron model A over R may be obtained as the quotient P/E, where $P \subset \mathrm{Pic}_{X/R}$ is the open subgroup scheme consisting of line bundles with total degree zero on fibers of $X \to \mathrm{Spec}\ R$, and E is the closure of the identity section of P_K (P/E is the maximal separated quotient of P). Then $\pi_0(P_s)$ can be identified with $L^\vee$ (by evaluating degrees along each component of X_s) and $E(R) \cong L$ (line bundles trivial on the generic fiber are of the form $\mathcal{O}(D)$ for divisors D supported on the special fiber), and the natural map $E(R) \to P(R) \to \pi_0(P_s)$ coincides with the map $L \to L^\vee$ given by the intersection pairing. □

5.5. **Proof of Prop 4.6.** Since $\mathrm{Pic}^\circ_{X/C} \cong A^\circ$, we have $\det(\mathbf{R}^1 f_* \mathcal{O}_X) \cong \omega^{-1}_{A/C}$. Applying Riemann-Roch to $\mathbf{R}^1 f_* \mathcal{O}_X$ we get

$$\chi(C, \mathbf{R}^1 f_* \mathcal{O}_X) = -\deg \omega_{A/C} - n(g-1).$$

Therefore

$$\alpha(X) = \chi(C, \mathcal{O}_C) - \chi(C, \mathbf{R}^1 f_* \mathcal{O}_X) - 1 + \dim \mathrm{Pic}^\circ_{X/k} = \deg \omega_{A/C} + n(g-1) + \dim \mathrm{Pic}^\circ_{X/k} - g.$$

Since $B \cong \mathrm{Pic}^\circ_{X/k}/\mathrm{Pic}^0_{C/k}$, we get $\dim \mathrm{Pic}^\circ_{X/k} - g = \dim B$, hence the formula (4.1). □

References

[1] Bosch, S.; Lütkebohmert, W.; Raynaud, M. *Néron models.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990.

[2] Conrad, B. Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. Enseign. Math. (2) 52 (2006), no. 1-2, 37-108.

[3] Gordon, W. J. Linking the conjectures of Artin–Tate and Birch–Swinnerton-Dyer. Compositio Math. 38 (1979), no. 2, 163-199.

[4] Grothendieck, A. Le groupe de Brauer. III. Exemples et compléments. In *Dix exposés sur la cohomologie des schémas*, 88-188, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1968.

[5] Raynaud, M. Spécialisation du foncteur de Picard. Publ. Math. IHES. 38 (1970), 27-76.

[6] Tate, J. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki, Vol. 9, Exp. No. 306, 415-440, Soc. Math. France, Paris, 1995.

[7] Ulmer, D. Curves and Jacobians over function fields. In *Arithmetic Geometry over Global Function Fields*, Advanced Courses in Mathematics CRM Barcelona (2014), 283-337.