Math 676. Homework 9

1. Read §7–§9 in the handout on absolute values.

2. Let p be a prime. Prove that the only elements of $\mathbf{Q}_p^\times$ that admit eth roots for all e relatively prime to p are the elements in $1+p\mathbf{Z}_p$, and deduce that all automorphisms of $\mathbf{Q}_p$ are continuous and hence equal the identity. Also prove that $\mathbf{Q}_p \not\simeq \mathbf{Q}_{p'}$ as abstract fields for any $p' \neq p$.

3. Let K be a (non-archimedean) local field with normalized absolute value $\|\cdot\|_K$. Let $\mathrm{d}x$ denote a Haar measure on K. Prove that $\mathrm{d}x/\|x\|_K$ on the open subset $K^\times$ is a Haar measure for $K^\times$. (Hint: $x \mapsto \|x\|_K$ is locally constant on $K^\times$)

4. Let F be the fraction field of a Dedekind domain A. Let $\mathfrak{m}$ be a maximal ideal of A and let $F_\mathfrak{m}$ be the corresponding completion of F, with valuation ring $A_\mathfrak{m}^\wedge$.

(*i*) If I and J are two fractional ideals of F, prove that $I = J$ in A if and only if $IA_\mathfrak{m}^\wedge = JA_\mathfrak{m}^\wedge$ inside of $F_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. (Hint: First reduce to the case when A is a discrete valuation ring.)

(*ii*) Let F'/F be a finite separable extension, and let A' be the integral closure of A in F' (so A' is a finite A-module). Let $\mathfrak{m}_i'$ $(1 \le i \le g)$ be the maximal ideals of A' over $\mathfrak{m}$, and let $F'_{\mathfrak{m}_i'}$ be the corresponding completion of F' (so it is a finite separable extension of $F_\mathfrak{m}$, as is shown in the study of completions in the handout). Let $A_{\mathfrak{m}_i'}^\wedge$ be the valuation ring of $F'_{\mathfrak{m}_i'}$.

Prove that $A_{\mathfrak{m}_i'}^\wedge$ is the integral closure of $A_\mathfrak{m}^\wedge$ in $F'_{\mathfrak{m}_i'}$, and that the nonzero product of local discriminants $\prod_i \mathfrak{d}_{A_{\mathfrak{m}_i'}^\wedge/A_\mathfrak{m}^\wedge}$ in $A_\mathfrak{m}^\wedge$ is equal to $\mathfrak{d}_{A'/A}A_\mathfrak{m}^\wedge$; in this sense, formation of the discriminant is "compatible" with completion. (Hint: reduce to the case when A is a discrete valuation ring.)

(*iii*) If $f \in F[X]$ is a monic separable polynomial with positive degree and F'/F is a splitting field, prove that a maximal ideal $\mathfrak{m}$ of A is unramified in F' if and only if the splitting field for f over $F_\mathfrak{m}$ is unramified.

5. Let K be a field that is complete with respect to a non-trivial non-archimedean absolute value $|\cdot|$.

(*i*) For $a_0, a_1, \dots \in K$, define $r = 1/\limsup |a_n|^{1/n} \in [0,\infty]$. Prove that $\sum a_n x^n$ converges for $x \in K$ if $|x| < r$ and does not converge if $|x| > r$. Prove that if it converges for one x_0 with $|x_0| = r$ then it converges for all x with $|x| = r$. Also prove that if $r > 0$ then $\sum a_n x^n = 0$ for all x near 0 only if $a_n = 0$ for all n.

(*ii*) Assume that K has characteristic 0 and that its residue field k has characteristic p. Replace $|\cdot|$ with a suitable power so that $|p| = 1/p$. Prove that the power series $\log_p(1+x) = \sum_{n\ge1}(-1)^{n+1}x^n/n$ converges if and only if $|x| < 1$, and that $\log_p((1+x)(1+y)) = \log_p(1+x+y+xy)$ is equal to $\log_p(1+x)+\log_p(1+y)$ for all $|x|, |y| < 1$. (Be rigorous!) Also prove that $|\log_p(1+x)| = |x|$ if $|x| < p^{-1/(p-1)}$.

(*iii*) Prove that $\mathrm{ord}_p(n!) = (n - S_n)/(p-1)$ for a positive integer n, where S_n is the sum of the digits in the ordinary base-p decimal expansion of n. Conclude that for K as in (*ii*), the formal power series $\exp_p(x) = \sum x^n/n!$ converges if and only if $|x| < p^{-1/(p-1)}$, and that $|\exp_p(x) - 1| = |x|$ for all such x. Prove that $\exp_p(x+y) = \exp_p(x)\exp_p(y)$ for $|x|, |y| < p^{-1/(p-1)}$.

(*iv*) Prove that open disc $|t-1| < r$ in K (as in (*ii*)) is a subgroup of $K^\times$ whenever $r < 1$, and show that $t \mapsto \log_p(t)$ and $x \mapsto \exp_p(x)$ define inverse isomorphism between the open disc $|t-1| < p^{-1/(p-1)}$ and the open disc $|x| < p^{-1/(p-1)}$.

(*v*) Taking $K = \mathbf{Q}_p$, prove that $\log_p$ maps the multiplicative group $1+p\mathbf{Z}_p$ homeomorphically onto the additive group $p\mathbf{Z}_p$ if $p > 2$ and that it maps $1+4\mathbf{Z}_2$ homeomorphically onto the additive group $4\mathbf{Z}_2$ if $p = 2$. Conclude that all elements of $1+8\mathbf{Z}_2$ have square roots in $\mathbf{Q}_2$, and as an application describe all quadratic extensions of $\mathbf{Q}_p$ for all primes p (the case $p = 2$ requires separate treatment).

6. (*i*) Prove that if K is a non-archimedean local field with residue characteristic p and $n \in \mathbf{Z}$ is nonzero, then $(K^\times)^n$ is open in $K^\times$ with finite index if $p \nmid n$ (hint: Hensel's lemma), and prove the same for $\mathrm{char}(K) = 0$ and *any* $n \in \mathbf{Z} - \{0\}$ by using the p-adic logarithm. Deduce that every subgroup of finite index in $K^\times$ is open if $\mathrm{char}(K) = 0$.

(*ii*) If K is a local field with positive characteristic p, then use the description of K as $k((t))$ to show that $(K^\times)^p$ is closed but not open in $K^\times$. Also prove the existence of a finite-index subgroup of $K^\times$ that is not open. (Hint: Prove that $K^\times/(K^\times)^p$ is an $\mathbf{F}_p$-vector space with uncountable dimension.)