Math 676. Homework 8

1. Read §1–§6 in the handout on absolute values.

2. Compute the initial terms of the p-adic expansion (up to a_4p^4) for $1/7$ and -3 in $\mathbf{Q}_p$ for $p = 2, 3, 5$, where the expansions are taken to be in the form $\sum a_n p^n$ with $a_n \in \mathbf{Z}$, $0 \le a_n < p$. Also compute a square root of 2 in $\mathbf{Q}_p$ for $p \in \{5, 7, 17\}$ up to the term a_4p^4, or explain why none exists; be clear on your choice of square root.

3. Let F be a field that is complete with respect to a non-trivial discretely-valued absolute value $|\cdot|$, and let A be the associated valuation ring (so A is a discrete valuation ring). Let F'/F be a finite extension (possibly inseparable!), and let A' be the valuation ring of F' with respect to the unique absolute value $|\cdot|'$ extending $|\cdot|$. Let k and k' be the respective residue fields of A and A'.

(*i*) Prove that if $\{a'_i\}$ is a finite set of elements of A' that have k-linearly independent images in k', then the a'_i's are F-linearly independent in F'. Conclude that $[k' : k] \le [F' : F]$, so $f = [k' : k]$ is *finite*. This is called the *residue field degree* attached to F'/F.

(*ii*) Using the norm-formula for $|\cdot|'$ in terms of $|\cdot|$, deduce that $|\cdot|'$ is discretely-valued, with $e \stackrel{\text{def}}{=} [|F'^{\times}| : |F^{\times}|] \le [F' : F]$, so A' is a discrete valuation ring and e is *finite*; this is called the *ramification degree* attached to F'/F. To justify the terminology, show that if $\pi \in A$ is a uniformizer and $\pi' \in A'$ is a uniformizer then $\pi = {\pi'}^e u'$ with $u' \in {A'}^{\times}$.

(*iii*) Let $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of A and A'. Choose $\{a'_i\}$ in A' lifting a k-basis of k'. Prove that the ef elements $a'_i{\pi'}^j$ for $1 \le i \le f$ and $0 \le j \le e-1$ are A-linearly independent, and use $\mathfrak{m}'$-adic completeness of A' and $\mathfrak{m}$-adic completeness of A to prove that the A-linear inclusion

$$\bigoplus_{1 \le i \le f, 0 \le j \le e-1} A a'_i {\pi'}^j \to A'$$

is an isomorphism (hint: first study $A'/{\mathfrak{m}'}^r$ for $1 \le r \le e$). Hence, A' is a *finite* A-module (so A' is the integral closure of A in F') and $[F' : F] = ef$.

4. Let K be a field equipped with a non-trivial non-archimedean absolute value $|\cdot|$. Let A be the valuation ring and $\mathfrak{m}$ its maximal ideal (so $\mathfrak{m} \ne 0$ since $|\cdot|$ is non-trivial).

(*i*) Prove that A is open in K and $\mathfrak{m}$ is open in A. Also prove that for any $c \in K^{\times}$, $\{x \in K \mid |x| \le |c|\}$ is open and closed, and is homeomorphic to A via $x \mapsto x/c$. Conclude that K is locally compact if and only if A is compact.

(*ii*) Assuming that A is compact, deduce that $\mathfrak{m}$ has *finite* index in A, and hence that the residue field $k = A/\mathfrak{m}$ is *finite*. Also prove that in such cases $|\cdot|$ must be discretely-valued (so A is a discrete valution ring) and *complete*. Finally, conclude in general that K is locally compact if and only if K is complete and k is finite.

(*iii*) Now assume K is locally compact and let $q = \#k$. Let the *normalized absolute value* $\|\cdot\|_K$ be the unique power of $|\cdot|$ with value group $q^{\mathbf{Z}}$ in $\mathbf{R}^{\times}$; that is, $\|\cdot\|_K = q^{-\mathrm{ord}_K}$ where $\mathrm{ord}_K : K^{\times} \twoheadrightarrow \mathbf{Z}$ is the normalized order function (sending uniformizers to 1). Prove that if μ is a Haar measure on K (which makes sense since K is locally compact) and $a \in K^{\times}$, then $\mu_a(S) = \mu(aS)$ (for Borel sets S) is a Haar measure on K and $\mu_a = \|a\|_K \cdot \mu$. In other words, the normalized absolute value computes the scaling effect by $K^{\times}$ on Haar measures of K.

5. Continuing with the preceding exercise, let K be field that is locally compact with respect to a non-trivial non-archimedean absolute value. Our aim is to classify all such K. (The archimedean case was handled in the handout on absolute values.) By Exercise 4, K is complete and its valuation ring $(A, \mathfrak{m})$ is a discrete valuation ring with residue field k that is finite, say with size q. Let p denote the characteristic of k.

(*i*) Assume $\mathrm{char}(K) > 0$. Prove that K must have characteristic p, and prove that the algebraic closure k_0 of $\mathbf{F}_p$ in K injects into k. Use Hensel's Lemma to prove that in fact $k_0 \to k$ is an isomorphism, so via the inverse (*Tecihmüller lifting*) we may canonically view k as a subfield of K (and hence A also has a canonical structure of k-algebra).

(*ii*) Continuing with (*i*), upon choosing a uniformizer π of A prove that there exists a unique continuous map of k-algebras $\phi_\pi : k[\![T]\!] \to A$ sending T to π (where $k[\![T]\!]$ is given its T-adic topology), and that this map is an *isomorphism* (Hint: use completeness for K to prove surjectivity). Conclude that there is an isometry $K \simeq k(\!(T)\!)$ over k when we use the normalized absolute values on both sides. Explain conversely why $k(\!(T)\!)$ with its T-adic topology is locally compact for any finite field, so this gives a classification (up to isomorphism) of local fields with positive characteristic. (It is a general theorem in commutative algebra that if $(A, \mathfrak{m})$ is *any* discrete valuation ring whose fraction field F is $\mathfrak{m}$-adically complete with positive characteristic then there exists an abstract isomorphism of rings $A \simeq k[\![T]\!]$ with k the residue field of A, but if k is not algebraic over $\mathbf{F}_p$ then this k-algebra structure on A is *not* canonical.)

(*iii*) Now assume K has characteristic 0, so $p \in \mathfrak{m}$ is nonzero. Let $e = \mathrm{ord}_K(p) > 0$ and let $f = [k : \mathbf{F}_p]$. Let π be a uniformizer of A. Explain why K is canonically an extension of $\mathbf{Q}_p$, with $\|\cdot\|_K$ restricting to $|\cdot|_p^{ef}$, and prove that if $\{a_i\}$ in A is a set with $\mathbf{F}_p$-linearly independent image in k then the a_i's are $\mathbf{Q}_p$-linearly independent in K.

(*iv*) Continuing with (*iii*), let $\{a_i\}$ in A lift an $\mathbf{F}_p$-basis of k. Prove that $\pi^e = pu$ with $u \in A^\times$, and deduce (via π-adic completeness of A and p-adic completeness of $\mathbf{Z}_p$) that the natural $\mathbf{Z}_p$-linear map

$$\bigoplus_{1 \le i \le f, 0 \le j \le e-1} a_i \pi^j \mathbf{Z}_p \to A$$

is an isomorphism. Conclude that $[K : \mathbf{Q}_p] = ef$ is *finite*. Conversely, prove that any finite extension of $\mathbf{Q}_p$ (endowed with its canonical topology using the absolute value extending that on $\mathbf{Q}_p$) is a locally compact field. Hence, the non-archimedean local fields with characteristic 0 are precisely the finite extensions of the p-adic fields $\mathbf{Q}_p$.