Math 676. Homework 10

1. Let K be a global function field; that is, a finitely generated extension of some $\mathbf{F}_p$ with transcendence degree 1. Let the finite field k denote the algebraic closure of $\mathbf{F}_p$ in K. (This is the *constant field* of K.) Since k is perfect, there exists a *separating transcendence basis*: an element $x \in K^\times$ transcendental over k such that K is finite separable over $k(x)$.

(*i*) Prove that the set M_K of topological equivalence classes of non-trivial absolute values on K is countable, and that for each such equivalence class v the associated valuation ring A_v is a discrete valuation ring with fraction field K and residue field $\kappa(v)$ that is finite. (Hint: Use an extension structure $K/k(x)$ to reduce to the case $K = k(x)$.) For each v, we write $|\cdot|_v$ to denote the unique representative for v whose value group is $q_v^{\mathbf{Z}}$, where $q_v = \#\kappa(v)$. Prove that for each $f \in K^\times$ we have $|f|_v = 1$ for all but finitely many $v \in M_K$, and prove that $|f|_v = 1$ for *all* $v \in M_K$ if and only if $f \in k^\times$; in other words, just like for number fields, $|f|_v = 1$ for all v if and only if f is a root of unity! (Hint: study a minimal polynomial for f over $k(x)$)

(*ii*) Prove the *product formula*: $\prod_v |f|_v = 1$ for all $f \in K^\times$. (Hint: Use an extension structure $K/k(x)$ to reduce to the case $K = k(x)$, imitating the method of reduction to $\mathbf{Q}$ in the number field case.)

(*iii*) Let S be a finite non-empty set in M_K. The *Riemann-Roch theorem* over k, applied to K/k, ensures that there exists a separating transcendence basis x for K/k such that $K/k(x)$ is not only finite and separable but even has S as exactly the set of places over the infinite place on $k(x)$.

The ring $\mathscr{O}_{K,S}$ of *S-integers* in K is the set of $f \in K$ such that $f \in A_v$ for all $v \notin S$. Use an x as above to prove that $\mathscr{O}_{K,S}$ is a Dedekind domain finitely generated as an k-algebra, and that its maximal ideals are in one-to-one correspondence with the places on K outside of S. Moreover, if a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_{K,S}$ corresponds to a place $v \notin S$, then prove that the algebraic localization $(\mathscr{O}_{K,S})_{\mathfrak{m}}$ inside of K is equal to the valuation ring A_v.

(*iv*) The *adele ring* $\mathbf{A}_K \subseteq \prod_v K_v$ is the directed union of subrings $\mathbf{A}_{K,S} = \prod_{v\in S} K_v \times \prod_{v \notin S} \mathscr{O}_v$. Using the evident locally compact Hausdorff topological ring structure on the $\mathbf{A}_{K,S}$'s, explain how to give $\mathbf{A}_K$ a structure of locally compact Hausdorff topological ring. Also prove that in general $\mathbf{A}_K^\times$ with its subspace topology is not open in $\mathbf{A}_K$ and is not a topological group. (hint: For each non-archimedean place v, let $x_v \in \mathbf{A}_K$ be the adele with v-coordinate π_v and v'-coordinate 1 for all $v' \neq v$. Prove $x_v \in \mathbf{A}_K^\times$ with $\{x_v\}$ converging to 1 in $\mathbf{A}_K$ (for any choice of enumeration of M_K) but $\{x_v^{-1}\}$ has no limit in $\mathbf{A}_K$.)

2. Let K'/K be a finite separable extension of global fields. (Separability can be dropped, but this requires more commutative algebra than we have developed.)

(*i*) Define a natural continuous map $\mathbf{A}_K \to \mathbf{A}_{K'}$ over $K \to K'$ that is a homeomorphism onto a closed subring. (Hint: Study places of K' over a fixed place of K.)

(*ii*) Prove that the natural map of K'-algebras $K' \otimes_K \mathbf{A}_K \to \mathbf{A}_{K'}$ is a topological isomorphism, where the left side is given the product topology upon using any K-basis of K'. (Why does this latter topology not depend on the choice of K-basis of K'?)

(*iii*) Use (*ii*) to prove that K' is discrete in $\mathbf{A}_{K'}$ with compact quotient for any global field K' by reduction to the special cases $K' = \mathbf{Q}$ and $K' = k(x)$ for a finite field k.

(*iv*) If R is a topological ring, show that giving $R^\times$ the subspace topology from $R \times R$ via the identification $x \mapsto (x, x^{-1})$ onto the subset $\{(x,y) \in R^2 \mid xy = 1\}$ gives $R^\times$ a structure of topological group that is functorial in R. Applying this with $R = \mathbf{A}_K$ to give $\mathbf{A}_K^\times$ a structure of topological group (this is the *only* topology ever put on $\mathbf{A}_K^\times$), use discreteness of K in $\mathbf{A}_K$ to infer discreteness of $K^\times$ in $\mathbf{A}_K^\times$ and use openness of $\mathbf{A}_{K,S}$ in $\mathbf{A}_K$ to infer openness of $\mathbf{A}_{K,S}^\times$ in $\mathbf{A}_K^\times$. Show that each subset $\mathbf{A}_{K,S}^\times \subseteq \mathbf{A}_K^\times$ is given a product topology, and use this to describe a base of opens around the identity in $\mathbf{A}_K^\times$. The topological group $\mathbf{A}_K^\times$ is the *idele group* of K.

3. The purpose of this problem is to extend Galois theory to the case of infinite extensions. Recall that if K/k is an algebraic extension of fields then it is *separable* if all elements of K are separable over k, or equivalently if all intermediate fields of finite degree over k are separable over k, and it is *Galois* if every irreducible $f \in k[T]$ with a root in K (so f is separable) splits over K; equivalently, every finite subextension of K is contained in a Galois subextension. If K/k is Galois, we define $\mathrm{Gal}(K/k)$ to be $\mathrm{Aut}(K/k)$.

(*i*) Let k_s/k be a separable closure. Using the uniqueness of separable closure up to (non-unique) automorphism, prove that K/k is Galois if and only if K/k is separable and every k-embedding $K \hookrightarrow k_s$ has the same image.

(*ii*) Assume that K/k is Galois, and let K' be an intermediate extension (so K'/k is separable). Prove that K/K' is Galois and that K' is the fixed field of $\mathrm{Gal}(K/K')$ acting on K (hint: use (*i*) and uniqueness of separable closures up to isomorphism), and prove that K'/k is Galois if and only if $\mathrm{Gal}(K/K')$ is a normal subgroup of $\mathrm{Gal}(K/k)$, in which case the natural map of abstract groups $\mathrm{Gal}(K/k)/\mathrm{Gal}(K/K') \to \mathrm{Gal}(K'/k)$ is an isomorphism.

(*iii*) Assume K/k is Galois, and let Σ denote the set of subgroups of $\mathrm{Gal}(K/k)$ that arise in the form $\mathrm{Gal}(K/K')$ for intermediate extensions K'. By (*ii*), $K' \mapsto \mathrm{Gal}(K/K')$ is a bijection from the set of intermediate extensions to the set Σ, with K-Galois subextensions corresponding to normal subgroups in Σ, and that $H \mapsto K^H$ is the inverse bijection.

Prove as follows that Σ is not generally the set of *all* subgroups of $\mathrm{Gal}(K/k)$. Let k be finite of size q, with K an algebraic closure and k_n the unique subfield of degree n over k. Prove that the infinite cyclic group $\langle\phi\rangle \subseteq \mathrm{Gal}(K/k)$ generated by the qth-power map ϕ has fixed field k, yet this infinite cyclic group is *not* all of $\mathrm{Gal}(K/k)$: there exists a unique k-automorphism σ of K such that $\sigma|_{k_n}$ satisfies $\sigma(x) = x^{q^{e_n}}$ with $e_n = 1! + 2! + \cdots + (n-1)!$ for all $n \geq 1$, and $\sigma \notin \langle\phi\rangle$.

(*iv*) We define the *Krull topology* on $G = \mathrm{Gal}(K/k)$ as follows: a base of opens around σ is given by the subsets $U_F(\sigma) = \{g \in G \,|\, \sigma|_F = g|_F\}$ for subextensions F of finite degree over k. (That is, an element is "close" to σ if it agrees with σ on a large finite set of elements of K.) Prove that the subsets $U_F(\sigma)$ satisfy the axioms to be a base of opens for a topology on G, called the *Krull topology*, and that this induces exactly the subspace topology on G via the inclusion $G \subseteq \prod_F \mathrm{Gal}(F/k)$ as F ranges over the k-finite subextensions that are *Galois* over k and each finite group $\mathrm{Gal}(F/k)$ is given the discrete topology. (For example, if $[K:k]$ is finite then this gives the discrete topology to $\mathrm{Gal}(K/k)$.) Also prove that if $k_1 \to k_2$ is a map of fields and $K_1 \to K_2$ is a map of Galois extensions over $k_1 \to k_2$ then the induced map $\mathrm{Gal}(K_2/k_2) \to \mathrm{Gal}(K_1/k_1)$ is continuous; in particular, the Krull topology is *functorial*.

(*v*) Prove that $G = \mathrm{Gal}(K/k)$ with its Krull topology is a topological group, and prove that G is closed in $\prod_F \mathrm{Gal}(F/k)$. (hint: Prove G is the set of tuples $(g_F)_F$ satisfying the collection of conditions $g_{F_1}|_{F_2} = g_{F_2}$ for all pairs F_1 and F_2 with $F_2 \subseteq F_1$.) Conclude that the Krull topology makes G *compact* and Hausdorff, and use this to prove that if K' is an intermediate extension then the natural injection $\mathrm{Gal}(K/K') \to \mathrm{Gal}(K/k)$ is a homeomorphism onto a closed subgroup and for K'/k Galois the natural map $\mathrm{Gal}(K/k)/\mathrm{Gal}(K/K') \to \mathrm{Gal}(K'/k)$ is an isomorphism of topological groups (using the quotient topology on the source).

(*vi*) Prove that the closure of a subgroup H of a topological group G is also a subgroup (hint: for $h \in H$, prove $h \cdot \overline{H} = \overline{H} = \overline{H} \cdot h$, so $H \cdot \overline{h} \subseteq \overline{H}$ and $\overline{h} \cdot H \subseteq \overline{H}$ for all $\overline{h} \in \overline{H}$), and that if $H \subseteq \mathrm{Gal}(K/k)$ is a subgroup then $\mathrm{Gal}(K/K^H)$ is the closure of H with respect to the Krull topology. (hint: Use *finite* Galois theory to show that H surjects onto $\mathrm{Gal}(K'/K^H)$ for all subextensions K' that are finite Galois over K^H!) Deduce that the set Σ in (*iii*) is exactly the set of *closed* subgroups with respect to the Krull topology, so the Galois correspondence is rescued if we restrict attention to closed subgroups of G.