Math 676. Higher ramification groups

Let K be complete with respect to a non-trivial discrete valuation, and let K'/K be a finite Galois extension with Galois group G. We assume that the residue field k of K has characteristic $p > 0$, and we let k' be the residue field of K'. For $i \geq -1$, we define the *ith ramification group* to be

$$G_i = \{\sigma \in G \mid \sigma(\alpha) \equiv \alpha \bmod \mathfrak{m}_{K'}^{i+1} \text{ for all } \alpha \in \mathscr{O}_{K'}\}.$$

For example, $G_{-1} = G$ and $G_0 = I$ is the inertia subgroup (indeed, the equality $G_0 = I$ is just a restatement of the definition of I). Recall that the *wild inertia subgroup* $P \subseteq I$ is $\mathrm{Gal}(K'/K_{\mathrm{t}})$, where K_{t} is the maximal tamely ramified subextension. In particular, P is normal in I and is its unique p-Sylow subgroup (so P is normal in G, as I is normal in G).

Observe that

$$G_i = \ker(G \to \mathrm{Aut}(\mathscr{O}_{K'}/\mathfrak{m}_{K'}^{i+1})),$$

so the G_i's are a decreasing chain of normal subgroups of the finite group G. Also, we have $G_i = \{1\}$ for sufficiently large i. Indeed, the decreasing chain $\{G_i\}$ must eventually stabilize, and that stable "value" is the intersection of all of these subgroups. However, if $\sigma \in \cap_i G_i$ then for all $\alpha \in \mathscr{O}_{K'}$ we have $\sigma(\alpha) - \alpha \in \cap_{i \geq -1} \mathfrak{m}_{K'}^{i+1} = \{0\}$, so $\sigma(\alpha) = \alpha$ for all $\alpha \in \mathscr{O}_{K'}$. This forces σ to be the identity on the fraction field K' of $\mathscr{O}_{K'}$. Beware that it can happen that $G_i = G_{i+1}$ for many i. The specific i's for which G_i "changes" are subtle invariants of the extension K'/K.

The aim of this handout is to provide proofs of some of the basic properties of these ramification groups; for the whole story (especially the behavior with respect to quotients and the ever-mystifying "upper numbering" filtration), see the discussion of ramification theory in Serre's *Local Fields* and Chapter 1 in the book edited by Cassels and Frohlich. This is very important in the theory of conductors and "bad Euler factors" in L-functions.

Now assume k'/k is separable. The general strategy of the analysis will be to construct a canonical injection of the finite group G_0/G_1 into $k'^{\times}$ and for $i \geq 1$ to inject the finite group G_i/G_{i+1} into k' (or more canonically, into the 1-dimensional k'-vector space $\mathfrak{m}_{K'}^i/\mathfrak{m}_{K'}^{i+1}$). Since k' is a field of characteristic $p > 0$, the group $k'^{\times}$ has no non-trivial elements of p-power order, and so G_0/G_1 has order prime to p. Moreover, since all finite subgroups of the multiplicative group of a field are cyclic, G_0/G_1 is necessarily even a cyclic group of order prime to p. Since k' is an abelian group that is killed by p, it will likewise follow that the quotients G_i/G_{i+1} for $i \geq 1$ are finite abelian groups killed by p. The triviality of G_j for large j therefore implies that G_i is a p-group for $i \geq 1$, so we conclude that the normal subgroup $G_1 \subseteq G_0$ is a p-group, and so since G_0/G_1 has order prime to p it follows that G_1 is a p-Sylow subgroup of G_0 (and in fact is the unique one). In particular, $G_1 = P = \mathrm{Gal}(K'/K_{\mathrm{t}})$ is the wild inertia group.

1. The quotient G_0/G_1

For $\sigma \in G$ and a uniformizer π' of K', $\sigma(\pi')$ is another uniformizer and hence the ratio $\sigma(\pi')/\pi'$ is a unit. The key to the whole story is the observation that if $\sigma \in G_0$ then the element $\sigma(\pi')/\pi' \bmod \mathfrak{m}_{K'}$ in $k'^{\times}$ is independent of π'. Indeed, if $\widetilde{\pi}'$ is another uniformizer then $\widetilde{\pi}' = u\pi$ for a unit u, and so $\sigma(\widetilde{\pi}')/\widetilde{\pi}' = (\sigma(u)/u) \cdot (\sigma(\pi)/\pi)$ in $\mathscr{O}_{K'}$. Passing to k', independence of π' is exactly the statement $\sigma(u)/u \equiv 1 \bmod \mathfrak{m}_{K'}$ for all $u \in \mathscr{O}_{K'}^{\times}$, or in other words $\sigma(u) \equiv u \bmod \mathfrak{m}_{K'}$ for all $u \in \mathscr{O}_{K'}^{\times}$. This condition holds because $\sigma \in G_0$.

We now choose $\sigma \in G_0$ and consider the canonical map of sets $\phi_0 : G_0 \to k'^{\times}$ defined by $\phi_0(\sigma) = \sigma(\pi')/\pi' \bmod \mathfrak{m}_{K'}$ for any choice of uniformizer π' (the choice of which does not matter, as we have just seen).

Theorem 1.1. *The map ϕ_0 is a homomorphism and its kernel is G_1. In particular, the finite group G_0/G_1 canonically injects into $k'^{\times}$ and so it has order prime to p (as k' is a field of characteristic $p > 0$ and hence contains no non-trivial pth roots of unity).*

Proof. Due to the independence of π' in the definition of ϕ_0, for $\sigma, \tau \in G$ we compute

$$\frac{(\sigma\tau)(\pi')}{\pi'} = \frac{\sigma(\tau(\pi'))}{\tau(\pi')} \cdot \frac{\tau(\pi')}{\pi'},$$

so since $\tau(\pi')$ is a uniformizer we conclude by reducing modulo $\mathfrak{m}_{K'}$ that $\phi_0(\sigma\tau) = \phi_0(\sigma)\phi_0(\tau)$. Hence, ϕ_0 is a homomorphism.

By definition, $\sigma \in \ker(\phi_0)$ if and only if $\sigma(\pi')/\pi' \equiv 1 \bmod \mathfrak{m}_{K'}$, and multiplying by π' gives the equivalent statement $\sigma(\pi') \equiv \pi' \bmod \mathfrak{m}_{K'}^2$. Hence, G_1 certainly lies in $\ker(\phi_0)$. Thus, we have a map of groups $G_0/G_1 \to k'^{\times}$ induced by ϕ_0. We need to show that this is injective. That is, we pick $\sigma \in G_0$ such that $\sigma(\pi') \equiv \pi' \bmod \mathfrak{m}_{K'}^2$ for some (in fact, every) uniformizer π', and we seek to show $\sigma(\alpha) \equiv \alpha \bmod \mathfrak{m}_{K'}^2$ for all $\alpha \in \mathscr{O}_{K'}$. For this we have to use the hypothesis that k'/k is separable.

More specifically, since G_0 is the inertia subgroup, its fixed field K_{un} has residue field that is the separable closure of k in k' and so is equal to k'. In other words, $\mathscr{O}_{K_{\mathrm{un}}}$ surjects onto k'. Hence, we can use elements of $\mathscr{O}_{K_{\mathrm{un}}}$ as a "digit set" for π'-adic expansions. In particular, for any $\alpha \in \mathscr{O}_{K'}$ there exist $c_0, c_1 \in \mathscr{O}_{K_{\mathrm{un}}}$ such that $\alpha \equiv c_0 + c_1\pi' \bmod \mathfrak{m}_{K'}^2$. Since c_0 and c_1 are G_0-*invariant* and $\phi_0(\sigma) = 1$, we have

$$\sigma(\alpha) \equiv c_0 + c_1\sigma(\pi') \equiv c_0 + c_1\pi' \equiv \alpha \bmod \mathfrak{m}_{K'}^2.$$

Hence, indeed $\sigma \in G_1$ as desired. ■

2. The quotient G_i/G_{i+1} for $i \geq 1$

Now we turn our attention to the higher subquotients G_i/G_{i+1} for $i \geq 1$. Since $G_i \subseteq G_0$, for all $\sigma \in G_i$ the element $\sigma(\pi')/\pi' \in k'^{\times}$ is again independent of π'. By definition of G_i, we have $\sigma(\pi') \equiv \pi' \bmod \mathfrak{m}_{K'}^{i+1}$, so $\sigma(\pi')/\pi' \equiv 1 \bmod \mathfrak{m}_{K'}^{i}$. We will be interested in the residue class

$$\frac{\sigma(\pi')}{\pi'} - 1 \bmod \mathfrak{m}_{K'}^{i+1}$$

that lies in $\mathfrak{m}_{K'}^{i}/\mathfrak{m}_{K'}^{i+1}$.

Lemma 2.1. *For $\sigma \in G_i$,*

$$\phi_i(\sigma) = \frac{\sigma(\pi')}{\pi'} - 1 \bmod \mathfrak{m}_{K'}^{i+1} \in \mathfrak{m}_{K'}^{i}/\mathfrak{m}_{K'}^{i+1}$$

is independent of π'.

Proof. Much as in the case $i = 0$, for independence of the choice of π' we now have to show that if u is a unit then $\sigma(u)/u \equiv 1 \bmod \mathfrak{m}_{K'}^{i+1}$. This is equivalent to the condition $\sigma(u) \equiv u \bmod \mathfrak{m}_{K'}^{i+1}$ for all $u \in \mathscr{O}_{K'}^{\times}$, and this latter condition holds because $\sigma \in G_i$. ■

The map of sets $\phi_i : G_i \to \mathfrak{m}_{K'}^{i}/\mathfrak{m}_{K'}^{i+1}$ as defined in the lemma is the analogue of the map $\phi_0 : G_0 \to k'^{\times}$ that was studied above.

Theorem 2.2. *The map ϕ_i is a homomorphism of groups and $\ker(\phi_i) = G_{i+1}$. In particular, the finite group G_i/G_{i+1} injects into a 1-dimensional k'-vector space and so it is an abelian group killed by p.*

Proof. The lemma ensures that the definition of ϕ_i is independent of the choice of π'. To show $\phi_i(\sigma\tau) = \phi_i(\sigma)\phi_i(\tau)$ for $\sigma, \tau \in G_i$, first note that since k' is the residue field of K_{un} and $\tau \in G_i$ we have $\tau(\pi')/\pi' \equiv 1 + c_\tau {\pi'}^i \bmod \mathfrak{m}_{K'}^{i+1}$ for some $c_\tau \in \mathscr{O}_{K_{\mathrm{un}}}$. That is, $\tau(\pi') \equiv \pi' + c_\tau {\pi'}^{i+1} \bmod \mathfrak{m}_{K'}^{i+2}$. Similarly, $\sigma(\pi') \equiv \pi' + c_\sigma {\pi'}^{i+1} \bmod \mathfrak{m}_{K'}^{i+2}$ for some $c_\sigma \in \mathscr{O}_{K_{\mathrm{un}}}$. We apply σ to the congruence for $\tau(\pi')$ and use that $c_\tau \in \mathscr{O}_{K_{\mathrm{un}}}$ is fixed by the subgroup G_0 that contains G_i to infer

$$(\sigma\tau)(\pi') \equiv \sigma(\pi') + c_\tau(\sigma(\pi'))^{i+1} \equiv \pi' + c_\sigma {\pi'}^{i+1} + c_\tau(\pi' + c_\sigma {\pi'}^{i+1})^{i+1} \bmod \mathfrak{m}_{K'}^{i+2},$$

and the final term is the same as ${\pi'}^{i+1}$ modulo $\mathfrak{m}_{K'}^{i+2}$ since all other terms in the binomial expansion are divisible by ${\pi'}^{i+2}$ (even the final term ${\pi'}^{(i+1)^2}$, with $(i+1)^2 \geq i+2$ since $i \geq 1$!). Hence,

$$(\sigma\tau)(\pi') \equiv \pi' + (c_\sigma + c_\tau){\pi'}^{i+1} \bmod \mathfrak{m}_{K'}^{i+2},$$

so dividing by π' and subtracting 1 gives

$$\phi_i(\sigma\tau) = c_\sigma + c_\tau = \phi_i(\sigma) + \phi_i(\tau)$$

in $\mathfrak{m}_{K'}^{i}/\mathfrak{m}_{K'}^{i+1}$. ■