Math 676. Surjectivity for inertia groups

Let F be complete for a non-trivial discretely-valued place, and let $F''/F'/F$ be a tower of finite extensions with F'' and F' Galois over F. We give F' and F'' their unique places extending the one on F, and so inside of $\mathrm{Gal}(F''/F)$ and $\mathrm{Gal}(F'/F)$ there are (by definition) inertia subgroups $I(F''/F)$ and $I(F'/F)$ that are the respective kernels

$$I(F''/F) = \ker(\mathrm{Gal}(F''/F) \to \mathrm{Aut}(k''/k)), \quad I(F'/F) = \ker(\mathrm{Gal}(F'/F) \to \mathrm{Aut}(k'/k)).$$

Thus, the natural quotient map $\mathrm{Gal}(F''/F) \twoheadrightarrow \mathrm{Gal}(F'/F)$ carries $I(F''/F)$ into $I(F'/F)$. We wish to show that this map of inertia groups is also surjective.

The separable closures of k in k' and k'' are the residue fields of the maximal unramified subfields F'_{u} and F''_{u} of F in F' and F'' respectively, and so the Galois property of these fields over F (why are they Galois?) implies that these respectively separable closures are Galois over k. Hence, k'/k and k''/k are normal extensions, and the fixed fields of the inertia subgroups are the respective maximal unramified subextensions (why?). Since F''_{u} certainly contains F'_{u}, we may replace F with F'_{u} to reduce to the case when $I(F'/F) = \mathrm{Gal}(F'/F)$ (that is, F'/F has trivial maximal unramified subextension). In this case we need to show that $I(F''/F) = \mathrm{Gal}(F''/F''_{\mathrm{u}})$ surjects onto $\mathrm{Gal}(F'/F)$. Since F''_{u}/F is unramified and Galois yet F'/F is separable with trivial maximal unramified subextension, it follows that F''_{u} and F' are linearly disjoint over F. In other words, the compositum $F''_{\mathrm{u}}F'$ inside of F'' is identified with $F''_{\mathrm{u}} \otimes_F F'$ and so the natural injective map

$$\mathrm{Gal}(F''_{\mathrm{u}}F'/F''_{\mathrm{u}}) \hookrightarrow \mathrm{Gal}(F'/F)$$

is an isomorphism. However, the composite map

$$I(F''/F) = \mathrm{Gal}(F''/F''_{\mathrm{u}}) \twoheadrightarrow \mathrm{Gal}(F''_{\mathrm{u}}F'/F''_{\mathrm{u}}) \simeq \mathrm{Gal}(F'/F)$$

is exactly the canonical map being considered above, and so its surjectivity is now proved.

In general, if F_{s}/F is a separable closure and we choose an F-embedding of F' into F_{s}, then we use the unique extension of the place on v to a (not discretely-valued nor complete) place on F_{s} to define the inertia subgroup I_F in $G_F = \mathrm{Gal}(F_{\mathrm{s}}/F)$: it is the subgroup of elements that act trivially on the residue field at this place of F_{s}. It is clear (check!) that $I_F = \varprojlim I(F''/F)$ inside of $G_F = \varprojlim \mathrm{Gal}(F''/F)$ with F'' ranging over the finite Galois extensions of F inside of F_{s}. The preceding considerations imply that if F'/F is a fixed finite Galois extension and we fix an F-embedding of F' into F_{s} then $I(F''/F)$ surjects onto $I(F'/F) \subseteq \mathrm{Gal}(F'/F)$ for all F''/F' that are finite Galois over F inside of F_{s}. Hence, since an inverse limit of finite non-empty sets is non-empty (Zorn's Lemma), the natural map $I_F \to I(F'/F)$ must be surjective. That is, under the quotient map

$$G_F \twoheadrightarrow \mathrm{Gal}(F'/F)$$

we have $I_F \twoheadrightarrow I(F'/F)$.

If we replace "unramified" with "tamely ramified" in everything that went before, and so replace inertia groups $I(F'/F)$ and I_F with wild inertia subgroups $P(F'/F)$ and P_F (and likewise use maximal tamely ramified subextensions to replace the role of maximal unramified subextensions) then everything goes through exactly the same way: $P(F''/F) \to P(F'/F)$ is surjective for towers $F''/F'/F$ with F'' and F' finite Galois over F, $P_F = \varprojlim P(F'/F)$ inside of G_F, and the canonical map $P_F \to P(F'/F)$ is surjective for all finite Galois extensions F'/F inside of F_{s}. In this way, the inertia and wild inertia groups have good analogues for F_{s}/F that are well-behaved with respect to passage to finite level. The same works if we replace F_{s} with any Galois extension of F (with possibly infinite degree), since all that was used about F_{s} above was that it is Galois over F.

For infinite-degree analogues of the higher ramification groups, the situation is rather more subtle and is bound up with the story of "upper numbering" that is well-explained in Serre's book *Local fields*.

In number theory, the case of most interest for the preceding considerations is that of non-archimedean local fields. For Galois towers of global fields (or possibly infinite-degree Galois extensions of global fields) we thereby get corresponding surjectivity statements for inertia and wild inertia subgroups at non-archimedean places (upon identifying the associated decomposition groups with Galois groups over a local field).