# Math 676. Existence of discriminant ideal

## 1. Introduction

Let $A$ be a *noetherian* domain and $B$ an $A$-algebra that is finitely generated and torsion-free as an $A$-module. We do *not* assume $B$ to be a domain because our later applications in the context of completions of number fields will naturally give rise to examples for which $B$ is not a domain. For any multiplicative set $S$ of $A$ not containing 0, the $S^{-1}A$-module $S^{-1}B$ has a unique structure of $S^{-1}A$-algebra such that the natural map $B \to S^{-1}B$ is a map of rings. Indeed, the only possibility is to define $(b/s)(b'/s') = bb'/ss'$, and the well-definedness (that is, the dependence only on the fractions $b/s$ and $b'/s'$ in $S^{-1}B$ rather than on the choices of representative numerators and denominators) is a straightforward exercise in "clearning denominators".

We assume that $B$ is "locally free" as an $A$-module in the sense that for every maximal ideal $\mathfrak{m}$ of $A$, the finitely generated $A_{\mathfrak{m}}$-module $B_{\mathfrak{m}}$ (the localization of $B$ at $A - \mathfrak{m}$) is a *free* $A_{\mathfrak{m}}$-module. In particular, we get a well-defined principal discriminant ideal $\mathrm{disc}(B_{\mathfrak{m}}/A_{\mathfrak{m}})$ of $A$, generated by $\det(\mathrm{Tr}_{B_{\mathfrak{m}}/A_{\mathfrak{m}}}(b_ib_j))$ for any choice of ordered $A_{\mathfrak{m}}$-basis $\{b_1, \dots, b_n\}$ of $B_{\mathfrak{m}}$ (such a determinant depends on the basis up to a unit square, and so it generates a principal ideal that is independent of the basis).

**Lemma 1.1.** *For any multiplicative set $S$ of $A$ (with $0 \notin S$), the $S^{-1}A$-algebra $S^{-1}B$ is finitely generated and torsion-free as an $S^{-1}A$-module, and is locally free as an $S^{-1}A$-module.*

*Proof.* It is clear that $S^{-1}B$ is finitely generated and torsion-free as an $S^{-1}A$-module. To verify local freeness, pick a maximal ideal $M$ of $S^{-1}A$, so $M$ "corresponds" to a prime ideal $\mathfrak{p}$ of $A$ not meeting $S$ (that is, $M = S^{-1}\mathfrak{p}$); beware that $\mathfrak{p}$ need *not* be maximal in $A$ (perhaps there exist strictly larger primes in $A$, but they all meet $S$). It is straightforward to identify $(S^{-1}A)_M$ with $A_{\mathfrak{p}}$ and $(S^{-1}B)_M$ with $B_{\mathfrak{p}}$, so we just have to show that $B_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of $A$. By Zorn's Lemma (or the noetherian property of $A$) there exists a maximal ideal $\mathfrak{m}$ of $A$ containing $\mathfrak{p}$, and by hypothesis $B_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module. Localization at the prime $\mathfrak{p}A_{\mathfrak{m}}$ then implies (using the same module basis) that $B_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. ∎

Our goal is to prove the existence and uniqueness of an ideal $\mathfrak{d}_{B/A}$ of $A$ such that the induced ideal $(\mathfrak{d}_{B/A})_{\mathfrak{m}} = \mathfrak{d}_{B/A} \cdot A_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ coincides with $\mathrm{disc}(B_{\mathfrak{m}}/A_{\mathfrak{m}})$ for all $\mathfrak{m}$. We also want some further properties to hold:

**Theorem 1.2.** *In the setup given above, the ideal $\mathfrak{d}_{B/A}$ does exist and is uniquely determined. Moreover, it satisfies the following properties:*

(1) *It is compatible with localization in the sense that for any multiplicative set $S$ of $A$, the localization $S^{-1}\mathfrak{d}_{B/A} = \mathfrak{d}_{B/A} \cdot S^{-1}A$ as an ideal of $S^{-1}A$ is equal to $\mathfrak{d}_{S^{-1}B/S^{-1}A}$.*
(2) *If $B$ is a free $A$-module, then $\mathfrak{d}_{B/A} = \mathrm{disc}(B/A)$ is the principal ideal generated by $\det(\mathrm{Tr}_{B/A}(b_ib_j))$ for an ordered $A$-basis $\{b_i\}$ of $B$.*

Note that Lemma 1.1 ensures that $\mathfrak{d}_{S^{-1}B/S^{-1}A}$ makes sense (once existence and uniqueness are proved in general). In fact, we also want an additional property to hold: compatibility with respect to extension of scalars to another noetherian domain. This will be essential in our later study of completions of number fields.

It should be emphasized at the outset that if one is only interested in Dedekind domains then the theory of the global theory of the discriminant can be developed by much more elementary methods (as you can find in any book on algebraic number theory). However, my feeling is that the elementary approach creates a misleading impression concerning what really makes the theory of the discriminant work: it is really a construction in linear algebra and has nothing at all to do with the theory of Dedekind domains. Hence, due to my stubbornness on this point, below I give a development that more fully brings out the "linear algebra" viewpoint on the theory. (If we had developed more powerful localization methods in commutative algebra, some of the argument below could be done differently and in much greater generality.)

## 2. Proof of Theorem 1.2

To prove uniqueness, we note that we are requiring that the ideal $\mathfrak{d}_{B/A}$ of $A$ have specified localizations at all maximal ideals of $A$. Thus, it suffices to apply the following lemma to the $A$-module $A$ and any two ideals of $A$ (considered as submodules of $A$).

**Lemma 2.1.** *Let $M$ be a torsion-free module over a ring $R$, and let $N, N' \subseteq M$ be two $R$-submodules. If $N_{\mathfrak{m}} \subseteq N'_{\mathfrak{m}}$ inside of $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $R$, then $N \subseteq N'$. The same holds for the condition of equality of submodules.*

*Proof.* The case of equality follows from two applications of the case of inclusion. Thus, it suffices to treat the case of inclusion. Pick $n \in N$ and let $I$ be the set of $r \in R$ such that $rn \in N'$ (such as $r = 0$). Clearly $I$ is an ideal of $R$; we want $I = R$, so $1 \in I$ (and hence $n \in N'$). If not, there exists a maximal ideal $\mathfrak{m}$ of $R$ containing $I$, and so by hypothesis $N_{\mathfrak{m}} \subseteq N'_{\mathfrak{m}}$. Thus, $n = n'/s$ for some $s \in A - \mathfrak{m}$ and $n' \in N'$, so $sn \in N'$. Hence, $s \in I$. However, $I \subseteq \mathfrak{m}$. This gives a contradiction. ■

With uniqueness established, we now turn to the proof that in the special case that $B$ is a free $A$-module, the principal ideal $\mathrm{disc}(B/A)$ satisfies the requirements and so must be the unique solution. Thus, we want to prove

$$\mathrm{disc}(B/A)A_{\mathfrak{m}} = \mathrm{disc}(B_{\mathfrak{m}}/A_{\mathfrak{m}})$$

for every maximal ideal $\mathfrak{m}$ of $A$. Let $\{b_i\}$ be an ordered $A$-basis of $B$, so clearly the $b_i$'s are also an $A_{\mathfrak{m}}$-basis of $B_{\mathfrak{m}}$. By definition of the two discriminant ideals (for $B$ over $A$ and for $B_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$), it suffices to prove the equality of determinants

$$\det(\mathrm{Tr}_{B/A}(b_ib_j)) = \det(\mathrm{Tr}_{B_{\mathfrak{m}}/A_{\mathfrak{m}}}(b_ib_j))$$

in $A_{\mathfrak{m}}$. More generally, we claim that if $S$ is any multiplicative set of $A$ (with $0 \notin S$) and $T : M \to M$ is any $A$-linear endomorphism of a finite free $A$-module (such as $M = B$ and $T$ equal to multiplication by an element of $B$ such as some $b_ib_j$) then there is an equality of traces $\mathrm{Tr}_A(T) = \mathrm{Tr}_{S^{-1}A}(T_S)$ in $S^{-1}A$ with $T_S : S^{-1}M \to S^{-1}M$ the $S^{-1}A$-linear endomorphism induced by $T$. This equality of traces is obvious: an $A$-basis of $M$ is an $S^{-1}A$-basis of $S^{-1}M$, and so $T$ and $T_S$ are each described by the *same* matrix via the use of such bases.

Let us now verify the compatibility with localization (assuming existence), by which we mean the following: if $\mathfrak{d}_{B/A}$ is known to exist then the ideal $S^{-1}\mathfrak{d}_{B/A}$ in $S^{-1}A$ satisfies the properties that uniquely characterize $\mathfrak{d}_{S^{-1}B/S^{-1}A}$. That is, for every maximal ideal $M$ of $S^{-1}A$ we claim that

$$(S^{-1}\mathfrak{d}_{B/A})_M \stackrel{?}{=} \mathrm{disc}((S^{-1}B)_M/(S^{-1}A)_M)$$

as ideals of $(S^{-1}A)_M$. Letting $\mathfrak{p}$ be the prime of $A$ not meeting $S$ with $\mathfrak{p} \cdot S^{-1}A = M$, we may restate the desired equality as

$$(\mathfrak{d}_{B/A})_{\mathfrak{p}} \stackrel{?}{=} \mathrm{disc}(B_{\mathfrak{p}}/A_{\mathfrak{p}}).$$

Choose a maximal ideal $\mathfrak{m}$ of $A$ containing $\mathfrak{p}$, so by the defining property of $\mathfrak{d}_{B/A}$ we have

$$(\mathfrak{d}_{B/A})_{\mathfrak{m}} = \mathrm{disc}(B_{\mathfrak{m}}/A_{\mathfrak{m}}) = \det(\mathrm{Tr}_{B_{\mathfrak{m}}/A_{\mathfrak{m}}}(b_ib_j))A_{\mathfrak{m}}$$

for an $A_{\mathfrak{m}}$-basis $\{b_i\}$ of $B_{\mathfrak{m}}$. Passing to the localization at $\mathfrak{p}A_{\mathfrak{m}}$ then gives

$$(\mathfrak{d}_{B/A})_{\mathfrak{p}} = \det(\mathrm{Tr}_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}(b_ib_j))A_{\mathfrak{p}}$$

in view of the compatibility of traces with respect to localization (as we noted above). The $b_i$'s are an $A_{\mathfrak{p}}$-basis of $B_{\mathfrak{p}}$, so we get the desired equality.

It remains to prove that $\mathfrak{d}_{B/A}$ exists! That is, we must construct an ideal $I$ of $A$ such that $I_{\mathfrak{m}} = \mathrm{disc}(B_{\mathfrak{m}}/A_{\mathfrak{m}})$ for every maximal ideal $\mathfrak{m}$ of $A$. As a first step, we need to "smear out" the local freeness hypothesis on $B$ as an $A$-module. For this purpose we finally (!) use the noetherian hypothesis on $A$ that has heretofore not been used.

**Lemma 2.2.** *Let $M$ be a torsion-free finitely generated module over a noetherian domain $A$. The $A$-module $M$ is locally free (in the sense that $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of $A$) if and only if there exist nonzero elements $a_1, \dots, a_n \in A$ generating $1$ (that is, $\sum a'_i a_i = 1$ for some $a'_i \in A$) such that the finitely generated $A[1/a_i]$-module $M[1/a_i]$ is free for all $i$.*

*Proof.* Assuming such $a_i$'s exist, each maximal ideal $\mathfrak{m}_0$ of $A$ must fail to contain some $a_{i_0}$ and hence the $A_{\mathfrak{m}_0}$-module $M_{\mathfrak{m}_0}$ is obtained by localization of the free $A[1/a_{i_0}]$-module $M[1/a_{i_0}]$ at the prime ideal $\mathfrak{m}_0 \cdot A[1/a_{i_0}]$. Localization preserves freeness, so the freeness after localization at $\mathfrak{m}_0$ follows.

Conversely, assuming that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-free for all $\mathfrak{m}$, we shall construct the desired $a_i$'s. In fact, for each $\mathfrak{m}$ we will construct an element $s_{\mathfrak{m}} \in A - \mathfrak{m}$ such that $M[1/s_{\mathfrak{m}}]$ is $A[1/s_{\mathfrak{m}}]$-free. The ideal generated by all $s_{\mathfrak{m}}$'s is not contained in any maximal ideal, and hence must be $A$. Thus, some $A$-linear combination of finitely many $s_{\mathfrak{m}}$'s gives 1, and these finitely many $s_{\mathfrak{m}}$'s can be taken to be the $a_i$'s. We may therefore now fix a choice of $\mathfrak{m}$ and focus on finding a single $s \in A - \mathfrak{m}$ such that $M[1/s]$ is $A[1/s]$-free, given that $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-free. The idea is to use "denominator-chasing". More specifically, since $A$ is *noetherian* we may use Homework 3, Exercise 5(*iii*), to see that the natural map

$$\mathrm{Hom}_A(N, N')_{\mathfrak{m}} \to \mathrm{Hom}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}, N'_{\mathfrak{m}})$$

is an isomorphism for any two finitely generated and torsion-free $A$-modules $N$ and $N'$. Hence, upon choosing an isomorphism $\phi : M_{\mathfrak{m}} \simeq A_{\mathfrak{m}}^{\oplus n}$ as $A_{\mathfrak{m}}$ we can find an $A$-linear map $T : M \to A^{\oplus n}$ and an $s \in A - \mathfrak{m}$ such that $s^{-1}T = \phi$. We can likewise find an $A$-linear map $T' : A^{\oplus n} \to M$ and an $s' \in A - \mathfrak{m}$ such that ${s'}^{-1}T' = \phi^{-1}$. Thus, over $A[1/ss']$ we have an $A[1/ss']$-linear isomorphism $s^{-1}T$ between $M[1/ss']$ and $A[1/ss']^{\oplus n}$. Since $ss' \in A - \mathfrak{m}$, we have the desired free localization of $M$ at an element outside of $\mathfrak{m}$. ∎

We now fix a choice of finitely many *nonzero* elements $a_1, \dots, a_n \in A$ generating 1 such that $B[1/a_i]$ is a free $A[1/a_i]$-module for all $i$. Thus, the ideal $I_i = \mathrm{disc}(B[1/a_i]/A[1/a_i])$ makes sense, and by compatibility of this "free-module" discriminant with respect to localization (which follows from compatibility of trace with respect to localization on the base ring) we have

$$I_i A[1/a_i a_j] = \mathrm{disc}(B[1/a_i a_j]/A[1/a_i a_j]) = I_j A[1/a_i a_j].$$

This equality of principal ideals encodes the fact that the determinant defining the discriminant in the free-module setting is well-defined up to unit multiple. We now glue ideals:

**Lemma 2.3.** *Let $R$ be a domain, and let $r_1, \dots, r_n \in R$ be nonzero elements that generate 1. Let $I_i$ be an ideal of $R[1/r_i]$ such that $I_i R[1/r_i r_j] = I_j R[1/r_i r_j]$ for all $i, j$. There exists a unique ideal $I$ of $R$ such that $IR[1/r_i] = I_i$ for every $i$.*

*Proof.* We first check uniqueness: if $\mathfrak{m}$ is a maximal ideal of $R$ then some $r_i \notin \mathfrak{m}$, and so $M = \mathfrak{m}R[1/r_i]$ is a prime (even maximal) ideal of $R[1/r_i]$. Clearly $I_{\mathfrak{m}} = (IR[1/r_i])_M = (I_i)_M$, so $I_{\mathfrak{m}}$ is uniquely determined for all $\mathfrak{m}$. Hence, $I$ is uniquely determined.

Now we check existence. Let $I = \cap I_j$ inside of the fraction field of $R$. We first claim that $I \subseteq R$, so $I$ is an ideal of $R$. Indeed, for any $x \in I$ we have $x \in I_j \subseteq R[1/r_j]$ for every $j$, so $r_j^{e_j} x \in R$ for some $e_j \geq 0$. Taking $e \geq e_j$ for all $j$, $r_j^e x \in R$ for all $j$. The $r_j$'s generate 1, so the $r_j^e$'s do too. Hence, if $\sum c_j r_j^e = 1$ with $c_j \in R$ then

$$x = 1 \cdot x = \sum c_j r_j^e x \in R.$$

We now need to check that $IR[1/r_i] = I_i$ for all $i$. It is clear that $IR[1/r_i] \subseteq I_i$ for all $i$. To get the reverse inclusion, choose $x_i \in I_i$. We seek a large integer $e$ such that $r_i^e x \in I$, for then $x_i = (r_i^e x_i)/r_i^e \in IR[1/r_i]$ as desired. Since $x_i \in I_i$ and $I_i[1/r_i r_j] = I_j[1/r_i r_j]$ for all $j$, we have $x_i = y_j/(r_i r_j)^{\mu_j}$ for some $y_j \in I_j$ and $\mu_j \geq 0$ for all $j$. Taking $e = \max_j \mu_j$, $r_i^e x_i \in I_j$ for all $j$ (since $I_j$ is an ideal of the ring $R[1/r_j]$). Hence, $r_i^e x_i \in I$. ∎

## 3. Behavior with respect to products and tensor products

We now want to explain some useful additional compatibility properties for discriminant ideals. The basic idea is always the same: we use localization to reduce to the case when various modules are free, and then we compare determinants. However, as we shall see, it sometimes require a bit of care to carry out this idea.

Let $A$ be a noetherian domain and let $B$ be an $A$-algebra that is finitely generated and torsion-free as an $A$-module, and assume that $B$ is also locally free as an $A$-module. If $B'$ is a second such $A$-algebra, then $B \times B'$ is also such an $A$-algebra. Thus, the discriminant ideals $\mathfrak{d}_{B\times B'/A}$, $\mathfrak{d}_{B/A}$, and and $\mathfrak{d}_{B'/A}$ make sense in $A$.

**Theorem 3.1.** *As ideals in $A$, $\mathfrak{d}_{B\times B'/A} = \mathfrak{d}_{B/A}\mathfrak{d}_{B'/A}$.*

*Proof.* This is just a calculation with "block matrices". Our problem is to compare two ideals of $A$, so it suffices to compare them in $A_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of $A$. In this case, we may rename $A_\mathfrak{m}$ as $A$ and acquire the extra property that $B$ and $B'$ are free as $A$-modules. Thus, the problem is to prove the equality of principal ideals

$$\mathrm{disc}(B \times B'/A) = \mathrm{disc}(B/A)\mathrm{disc}(B'/A).$$

We simply make an intelligent choice of $A$-bases: we pick $A$-bases $\mathbf{b} = \{b_i\}$ of $B$ and $\mathbf{b}' = \{b'_j\}$ of $B'$, and we take $\mathbf{e} = \{b_1, \dots, b_n, b'_1, \dots, b'_{n'}\}$ as the choice of $A$-basis of $B \times B'$. We claim that the equality of determinants

$$\det(\mathrm{Tr}_{B\times B'/A}(e_r e_s)) = \det(\mathrm{Tr}_{B/A}(b_i b_j))\det(\mathrm{Tr}_{B'/A}(b'_{i'} b'_{j'}))$$

holds in $A$ because $e_r e_s = 0$ unless $e_r, e_s \in \mathbf{b}$ or $e_r, e_s \in \mathbf{b}'$.

To be more precise, if $e_r = b_i$ and $e_s = b_j$ then

$$\mathrm{Tr}_{B\times B'/A}(e_r e_s) = \mathrm{Tr}_{B/A}(b_i b_j)$$

because the multiplication action by $e_r e_s = b_j b_j$ on $B \times B'$ preserves the two factors and kills the second one. A similar conclusion holds if $e_r, e_s \in \mathbf{b}'$, and so it follows that the matrix $(\mathrm{Tr}_{B\times B'/A}(e_r e_s))$ is a block matrix with upper-left block given by $(\mathrm{Tr}_{B/A}(b_i b_j))$ and lower-right block given by $(\mathrm{Tr}_{B/A}(b'_{i'} b'_{j'})$, and the off-diagonal blocks are equal to 0. The determinant of such a block matrix is the product of the determinants of the two blocks along the diagonal. ■

Another compatibility property for discriminants is behavior with respect to extension of scalars. Suppose that $A \to A'$ is a map between noetherian domains, and we allow the possibility that this map is not injective. The case of most interest to us will be when $A'$ is a "completion" of $A$, a certain kind of very large ring extension to be studied later. For any $A$-algebra $B$ that is torsion-free and finitely generated as an $A$-module, with $B$ even locally free as an $A$-module, consider the $A'$-algebra $B' = A' \otimes_A B$. We wish to compare the ideal $\mathfrak{d}_{B/A}$ of $A$ and the ideal $\mathfrak{d}_{B'/A'}$ of $A'$, but we first have to make sure that this latter ideal *makes sense*:

**Lemma 3.2.** *The $A'$-algebra $B'$ is finitely generated and torsion-free as an $A'$-module, with $B'$ locally free as an $A'$-module.*

*Proof.* If $B$ is generated by elements $b_1, \dots, b_n$ as an $A$-module, then $1 \otimes b_1, \dots, 1 \otimes b_n$ generate $B'$ as an $A'$-module. Thus, $B'$ is finitely generated over $A'$. To prove that $B'$ is torsion-free as an $A'$-module, we need to invoke a trick that was implicit in your solution to Exercise 5 ($v$) in Homework 3: in that solution you really proved that if a torsion-free and finitely generated module $M$ over a noetherian domain $R$ is locally free then $M$ is a direct summand of a free $R$-module $F$ of finite rank. (The Dedekind hypothesis was only relevant in your solution for the purpose of ensuring that each localization $M_\mathfrak{m}$ at a maximal ideal is a free module, so if such freeness is *assumed* then no Dedekind hypothesis is required!) One consequence of these considerations is that if $R \to R'$ is any map to a second noetherian domain then the $R'$-module $M' = R' \otimes_R M$ is a direct summand of a free $R'$-module of finite rank. Indeed, there is an isomorphism of $R$-modules $M \oplus N \simeq F$ with $F$ free of finite rank over $R$, and so by applying $R' \otimes_R (\cdot)$ and using the distributivity of tensor products over direct sums we get an isomorphism $M' \oplus N' \simeq F'$ with $F' = R' \otimes_R F$ free of finite rank over $R'$. Since $F'$ is visibly torsion-free as an $R'$-module (since $R'$ is a domain), the submodule $M'$ is also torsion-free.

Returning to our initial setup, $B' = A' \otimes_A B$ is finitely generated and torsion-free as an $A'$-module. We need to prove that it is even locally free as such. That is, if $\mathfrak{m}'$ is a maximal ideal of $A'$ then we want the finitely generated $A'_{\mathfrak{m}'}$-module $B'_{\mathfrak{m}'}$ to be free. The preimage $\mathfrak{p}$ of $\mathfrak{m}'$ in $A$ is a prime ideal (since $A/\mathfrak{p}$ is a subring of $A'/\mathfrak{m}'$ and hence is a domain), and $B_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module (since it may be obtained by first localizing at a maximal ideal $\mathfrak{m}$ of $A$ containing $\mathfrak{p}$, whereupon we get a free module over $A_{\mathfrak{m}}$, and such module-freeness is retained upon further localization at $\mathfrak{p}A_{\mathfrak{m}}$). Also note that the map $A \to A' \to A'_{\mathfrak{m}'}$ uniquely factors through $A \to A_{\mathfrak{p}}$. Indeed, uniqueness is clear because elements of $A - \mathfrak{p}$ have unit image in $A'_{\mathfrak{m}'}$ (even if the map $f : A \to A'$ is not injective), and for existence we just have to check that if $a_1, a_2 \in A$ and $s_1, s_2 \in A - \mathfrak{p}$ with $a_1/s_1 = a_2/s_2$ (that is $a_1 s_2 = a_2 s_1$ in $A$) then $f(a_1)/f(s_1) = f(a_2)/f(s_2)$ in $A'_{\mathfrak{m}'}$, but this is clear because $f(a_1)f(s_2) = f(a_1 s_2) = f(a_2 s_1) = f(a_2)f(s_1)$ in $A'$.) The exact same technique of proof shows that there is a unique map of rings $B_{\mathfrak{p}} \to B'_{\mathfrak{m}'}$ compatible with the map $B \to B'$ and the maps to the localizations (with localization taken for $B$ as a torsion-free $A$-module and $B'$ as a torsion-free $A'$-module, and these localizations are endowed with their evident unique well-defined ring structures as $B$-algebras and $B'$-algebras respectively). There is an evident natural $A'_{\mathfrak{m}'}$-module map

$$A'_{\mathfrak{m}'} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \to B'_{\mathfrak{m}'}$$

that is clearly a map of rings, and the left side is a free $A'_{\mathfrak{m}'}$-module since $B_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free. Thus, it suffices to show that this map is an isomorphism.

Rather more generally, if $M$ is any finitely generated and torsion-free $A$-module that is locally free, and if $M' = A' \otimes_A M$ (so $M'$ is also finitely generated and torsion-free over $A'$, as we proved above), then we claim that the natural $A'_{\mathfrak{m}'}$-linear map

$$\theta_M : A'_{\mathfrak{m}'} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \to M'_{\mathfrak{m}'}$$

(whose well-posedness is left as an instructive exercise) is an isomorphism. Surjectivity is obvious, but injectivity requires a trick as follows. Recall that the local freeness provides an isomorphism $M \oplus N \simeq F$ for a free $A$-module $F$ of finite rank. Note that $N$ is automatically finitely generated and torsion-free as an $A$-module (and it is also locally free, but we do not need this fact and so we ignore it). The formation of $\theta_M$ only requires knowing that $M$ is a direct summand of a free module (to ensure that $A' \otimes_A M$ is torsion-free over $A'$), and not that it is locally free, so $\theta_N$ makes sense too. It is obvious by definition-chasing that $\theta_M \oplus \theta_N = \theta_F$, so the injectivity of $\theta_M$ is reduced to that of $\theta_F$, and since $F$ a finite free module it is obvious (upon picking an $A$-basis of $F$) that $\theta_F$ is identified with a direct sum of finitely many copies of $\theta_A$. The map $\theta_A$ is clearly an isomorphism. ■

Now that it makes sense to form the discriminant ideal $\mathfrak{d}_{B'/A'}$, we can work out its relationship with $\mathfrak{d}_{B/A}$. All of the hard work was done in the proof of the preceding lemma.

**Theorem 3.3.** *As ideals of* $A'$, $\mathfrak{d}_{B/A}A' = \mathfrak{d}_{B'/A'}$.

The left side of the equality means the ideal of $A'$ generated by the image of $\mathfrak{d}_{B/A}$.

*Proof.* The problem is to compare two ideals of $A'$, so it suffices to compare them in $A'_{\mathfrak{m}'}$ for maximal ideals $\mathfrak{m}'$ of $A'$. Pick such an ideal, and let $\mathfrak{p}$ be its prime preimage in $A$. The right side of the desired equality localizes to the ideal $\mathrm{disc}(B'_{\mathfrak{m}'}/A'_{\mathfrak{m}'})$. The left side of the desired equality localizes to the ideal

$$(\mathfrak{d}_{B/A}A_{\mathfrak{p}})A'_{\mathfrak{m}'} = \mathfrak{d}_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}A'_{\mathfrak{m}'}$$

in $A'_{\mathfrak{m}'}$, since the formation of $\mathfrak{d}_{B/A}$ is compatible with localization at any multiplicative set of $A$ not containing 0 (such as $A - \mathfrak{p}$). Also recall from the proof of Lemma 3.2 that there is a natural isomorphism of $A'_{\mathfrak{m}'}$-algebras

$$B'_{\mathfrak{m}'} \simeq A'_{\mathfrak{m}'} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}},$$

so we may rename $A_{\mathfrak{p}}$, $B_{\mathfrak{p}}$, and $A'_{\mathfrak{m}'}$ as $A$, $B$, and $A'$ to reduce to the case when $B$ is a *free* $A$-module (and so $B' = A' \otimes_A B$ is a free $A'$-module).

Let $\{b_i\}$ be an $A$-module basis of $B$, so the elements $b'_i = 1 \otimes b_i$ are an $A'$-module basis of $B'$. Our problem is to prove the equality

$$(\det(\mathrm{Tr}_{B/A}(b_i b_j))A)A' = \det(\mathrm{Tr}_{B'/A'}(b'_i b'_j))A'$$

as ideals of $A'$, and so it suffices to prove that the map of rings $f : A \to A'$ carries the first determinant to the second. More specifically, since $b'_i b'_j = 1 \otimes (b_i b_j)$, it suffices to prove that $f(\mathrm{Tr}_{B/A}(b)) = \mathrm{Tr}_{B'/A'}(1 \otimes b)$ in $A'$ for any $b \in B$. The $A$-linear map $T : B \to B$ that is multiplication by $b$ induces the $A'$-linear map $T' : B' \to B'$ of multiplication by $1 \otimes b$ upon extension of scalars to $A'$, so we may now consider a more general problem: if $M$ is a finite free $A$-module and $T : M \to M$ is any $A$-linear endomorphism, we claim that $f(\mathrm{Tr}_A(T)) = \mathrm{Tr}_A(T')$, where $T' : M' \to M'$ is obtained by $T$ by extension of scalars. That is, we claim that the formation of trace is compatible with extension of scalars. This is obvious because an $A$-basis of $M$ induces an $A'$-basis of $M'$ and the matrices of $T$ and $T'$ with respect to such compatible bases are the "same" in the sense that $f : A \to A'$ carries the matrix of $T$ to the matrix of $T'$. ■

We now study discriminants for tensor-product algebras. As always, we let $A$ be a noetherian domain, but now we let $A_1$ and $A_2$ be two $A$-algebras that are finitely generated and torsion-free as $A$-modules, with each $A_i$ locally free as an $A$-module. We want to compute the ideal $\mathfrak{d}_{A_1 \otimes_A A_2/A}$ of $A$ in terms of the $\mathfrak{d}_{A_i/A}$'s, but we first have to prove that it even *makes sense* to contemplate a discriminant ideal for the tensor product algebra.

**Lemma 3.4.** *With notation and hypotheses as above, $A_1 \otimes_A A_2$ is a finitely generated and torsion-free $A$-module, and it is locally free. Moreover, for any maximal ideal $\mathfrak{m}$ of $A$ there is a natural isomorphism of $A_\mathfrak{m}$-algebras*

$$(A_1)_\mathfrak{m} \otimes_{A_\mathfrak{m}} (A_2)_\mathfrak{m} \to (A_1 \otimes_A A_2)_\mathfrak{m}.$$

*Proof.* More generally, if $M$ and $M'$ are finitely generated and torsion-free $A$-modules such that each is locally free as an $A$-module, then we claim that $M \otimes_A M'$ is a finitely generated and torsion-free $A$-module and that there is a natural map

$$\psi_{M,M'} : M_\mathfrak{m} \otimes_{A_\mathfrak{m}} M'_\mathfrak{m} \to (M \otimes_A M')_\mathfrak{m}$$

that is an isomorphism (and a map of rings if $M$ and $M'$ are $A$-algebras). Note that in particular it would follow that $M \otimes_A M'$ is locally free, since a tensor product of two free $A_\mathfrak{m}$-modules is a free $A_\mathfrak{m}$-module.

The same technique of "reduction to the case of free modules" as in the proof of Lemma 3.2 provides isomorphisms $M \oplus N \simeq F$ and $M' \oplus N' \simeq F'$ for finite free $A$-modules $F$ and $F'$, so $M \otimes_A M'$ is identified with a direct summand of the free modules $F \otimes_A F'$ and hence it is torsion-free. It is straightforward to define $\psi_{M,M'}$ in terms of bilinear pairings of fractions (one has to check well-definedness), and $\psi_{M,M'}$ is clearly a map of rings when $M$ and $M'$ are $A$-algebras. The map $\psi_{M,M'}$ is identified with a direct summand of the map $\psi_{F,F'}$, so for the isomorphism aspect it suffices to treat the case $M = F$ and $M' = F'$ because a direct summand of an isomorphism is an isomorphism. The case of free modules is trivial upon choosing a basis. ■

**Lemma 3.5.** *Let $M$ be a finitely generated and torsion-free $A$-module, and assume that $M$ is locally free. Let $r = \dim_F(F \otimes_A M)$. For every prime ideal $\mathfrak{p}$ of $A$, the free $A_\mathfrak{p}$-module $M_\mathfrak{p}$ has rank $r$.*

*Proof.* If we choose an $A_\mathfrak{p}$-basis of $M_\mathfrak{p}$, then further localization at all nonzero elements of $A_\mathfrak{p}$ yields $F \otimes_A M$ with the same elements now giving an $F$-basis (since a basis is the "same" as the data of an isomorphism to a standard free module). ■

For $M$ as in the preceding lemma, we call $r$ the *rank* of $M$ over $A$.

**Theorem 3.6.** *Let $A$ be a noetherian domain, and let $A_1$ and $A_2$ be $A$-algebras that are finitely generated and torsion-free as $A$-modules. If each $A_i$ is locally free as an $A$-module, with rank $n_i$, then*

$$\mathfrak{d}_{A_1 \otimes_A A_2/A} = \mathfrak{d}_{A_1/A}^{n_2} \mathfrak{d}_{A_2/A}^{n_1}.$$

As we shall see, this theorem is merely an elaborate version of the classical identity from tensor algebra $\det(T_1 \otimes T_2) = \det(T_1)^{d_2} \det(T_2)^{d_1}$ for linear endomorphisms $T_i$ of a $d_i$-dimensional vector space over a field.

*Proof.* In view of the two preceding lemmas, we may localize throughout at a maximal ideal of $A$ and hence we may assume that $A_1$ and $A_2$ are free $A$-modules with $n_i$ equal to the $A$-rank of $A_i$. We want $\mathrm{disc}(A_1 \otimes_A A_2/A) = \mathrm{disc}(A_1/A)^{n_2}\mathrm{disc}(A_2/A)^{n_1}$ as ideals of $A$. Let

$$\beta_i : A_i \times A_i \to A$$

be the symmetric $A$-bilinear trace pairing $\beta_i(a_i, a_i') = \mathrm{Tr}_{A_i/A}(a_i a_i')$. This induces a map of free $A$-modules

$$\beta_i' : A_i \to A_i^\vee = \mathrm{Hom}_A(A_i, A)$$

via $\beta_i'(a_i) = \beta_i(a_i, \cdot) = \beta_i(\cdot, a_i)$. Upon choosing a basis $\{e_r\}$ for $A_i$ and using its dual basis in $A_i^\vee$, the resulting matrix that describes $\beta_i'$ is *exactly* $(\beta_i(e_r e_{r'}))_{r,r'}$. Since a linear map $T : M \to N$ between two free $A$-modules of the same rank has a "determinant" $\det T \in A$ that is well-defined up to $A^\times$ (by using arbitrary choices of bases, or intrinsically by considering the annihilator of the cokernel of the induced map on top exterior powers), we conclude that $\det(\beta_i')A = \mathrm{disc}(A_i/A)$.

Now comes the key point. Via the natural isomorphism of free $A$-modules $A_1^\vee \otimes_A A_2^\vee \simeq (A_1 \otimes_A A_2)^\vee$ (which is defined by the condition $(\ell_1 \otimes \ell_2)(a_1 \otimes a_2) = \ell_1(a_1)\ell_2(a_2)$, we claim that the map $A_1 \otimes_A A_2 \to (A_1 \otimes_A A_2)^\vee$ arising from the trace form on $A_1 \otimes_A A_2$ is *exactly* $\beta_1 \otimes \beta_2$. By chasing evaluations on elementary tensors, this is just a fancy way to assert the general identity $\mathrm{Tr}_{A_1 \otimes_A A_2}(a_1 \otimes a_2) = \mathrm{Tr}_{A_1/A}(a_1)\mathrm{Tr}_{A_2/A}(a_2)$. To prove this latter identity, note that the multiplication map by $a_1 \otimes a_2$ on $A_1 \otimes_A A_2$ is $m_{a_1} \otimes m_{a_2}$ where $m_{a_i} : A_i \to A_i$ is multiplication by $a_i$. Hence, the asserted trace identity is a special case of the more general assertion that if $T : M \to M$ and $T' : M' \to M'$ are $A$-linear endomorphisms of finite free $A$-modules then the induced $A$-linear endomorphism $T \otimes T'$ of $M \otimes M'$ has trace $\mathrm{Tr}(T \otimes T')$ equal to $\mathrm{Tr}(T)\mathrm{Tr}(T')$. This is a "universal identity": it suffices to consider the special case when $M$ and $M'$ are standard free modules of respective ranks $r$ and $r'$ over the ring $\mathbf{Z}[X_{ij}, X'_{i'j'}]$ with $1 \le i, j \le r$ and $1 \le i', j' \le r'$, and $T$ and $T'$ are given by the matrices $(X_{ij})$ and $(X'_{i'j'})$ (and then a suitable map $\mathbf{Z}[X_{ij}, X'_{i'j'}] \to A$ will give the desired result in $A$). This universal case is over a domain as the base ring, and the verification in this case may be carried out by working over an algebraic closure $F$ of the fraction field of this base ring. A special feature of these "universal matrices" is that their characteristic polynomials have nonzero discriminants (due to the existence of *some* matrices with separable characteristic polynomial in every rank), and so over $F$ both matrices can be diagonalized. The trace identity in the case of diagonalized $T$ and $T'$ is trivial!

Returning to our original situation, we now have

$$\mathrm{disc}(A_1 \otimes_A A_2/A) = \det(\beta_1' \otimes \beta_2')A, \;\; \mathrm{disc}(A_1/A)^{n_2}\mathrm{disc}(A_2/A)^{n_1} = (\det(\beta_1')A)^{n_2}(\det(\beta_2')A)^{n_1}.$$

Hence, upon fixing $A$-bases of $A_1$ and $A_2$ and their duals (to identify all free modules with standard free modules) we are reduced to a determinant analogue of the above trace identity: if $T : M \to M$ and $T' : M' \to M'$ are $A$-linear endomorphisms of finite free $A$-modules with respectively ranks $r$ and $r'$, then $\det(T \otimes T') = \det(T)^{r'} \det(T')^r$. The exact same technique as used in the preceding paragraph reduces this universal identity to the special case of diagonalized endomorphisms over a field(with respect to suitable bases), and the diagonal case is a trivial calculation! ■

Another interesting question that can be asked concerning discriminant ideals is behavior with respect to towers of locally free ring extensions. This requires too much of a commutative algebra digression to discuss in the generality discussed above, but we will discuss it for Dedekind domains shortly.