Math 676. Dirichlet density for global fields

1. Motivation

Dirichlet introduced the functions $L(s,\chi)$ for Dirichlet characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ in order to study primes in arithmetic progressions. More specifically, he defined a notion of *Dirichlet density* for an arbitrary set of primes, and he proved that for each congruence class c in $(\mathbf{Z}/N\mathbf{Z})^\times$ the set of positive primes p such that $p \bmod N = c$ has a Dirichlet density and it is equal to $1/\phi(N) = 1/|(\mathbf{Z}/N\mathbf{Z})^\times|$. Roughly speaking, primes are "equally distributed" across congruence classes mod N, at least in the sense of Dirichlet density. In this classical setting, the Dirichlet density of a set Σ of positive primes is

$$\delta_{\mathrm{Dir}}(\Sigma) \stackrel{\mathrm{def}}{=} \lim_{s\to 1^+} \frac{\sum_{p\in\Sigma} p^{-s}}{\sum_p p^{-s}} = \lim_{s\to 1^+} \frac{\sum_{p\in\Sigma} p^{-s}}{\log(1/(s-1))}$$

if this limit exists; the final equality rests on the fact that $\zeta(s) \sim 1/(s-1)$ as $s \to 1^+$, which implies

$$\log(1/(s-1)) \sim \log\zeta(s) = \sum_p -\log(1-p^{-s}) \sim \sum_p p^{-s}$$

as $s \to 1$ because $\log(1/(s-1)) \to \infty$ yet upon expanding out $-\log(1-p^{-s}) = \sum_{j\geq 1} p^{-sj} = p^{-s} + \sum_{j\geq 2} p^{-sj}$ we see that the contribution $\sum_p \sum_{j\geq 2} p^{-sj}$ is bounded by $\zeta_{\mathbf{Q}}(2s)$ for real s near 1 (and even for real $s > 1/2$).

There is another notion of density that comes to mind, *natural density*:

$$\delta_{\mathrm{nat}}(\Sigma) \stackrel{\mathrm{def}}{=} \lim_{x\to\infty} \frac{\#\{p \leq x \,|\, p \in \Sigma\}}{\#\{p \leq x\}} = \frac{\#\{p \leq x \,|\, p \in \Sigma\}}{x/\log(x)},$$

using the prime number theorem for the final equality. It is a general fact that if Σ admits a natural density then it admits a Dirichlet density and these coincide. However, the converse is false: there are examples of sets of primes that admit a Dirichlet density but not a natural density; in this sense, Dirichlet density is a strictly more general notion (if perhaps a bit less intuitive than natural density). Dirichet's theorem is true in the sense of natural density, and this is a genuinely stronger assertion than the traditional form in terms of Dirichlet density. The main difference between the two concepts is that natural density rests on an ordering of the set of primes, whereas Dirichlet density does not.

For a positive prime p, the canonical identification of $(\mathbf{Z}/N\mathbf{Z})^\times$ with $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ carries $p \bmod N$ to the Frobenius element for $p\mathbf{Z}$. Thus, the generalization of Dirichlet's theorem beyond the classical case will concern representing elements of Galois groups (or rather, conjugacy classes in Galois groups) by Frobenius elements. If K'/K is a finite Galois extension of global fields, then all but finitely many non-archimedean places v of K are unramified in K' and so the Frobenius elements $\mathrm{Frob}(v'|v) \in G = \mathrm{Gal}(K'/K)$ make sense for places v' on K' over v. As we vary v', these Frobenius elements sweep out a conjugacy class in G, the *Frobenius conjugacy class of* v in G. (If G is abelian then conjugacy classes are elements and so it is well-posed to speak of a Frobenius element attached to a finite place v of K that is unramified in K'.) Hence, one is led to ask if a given conjugacy class c of G is a Frobenius conjugacy class for some non-archimedean place v of K that is unramified in K', and if so then "how often"? For example, can we say in a precise sense that the proportion of unramified v for which c is the Frobenius conjugacy class of v is $|c|/|G|$? This would be a good version of an "equidistribution" result for Frobenius classes in G as we vary v on K. To make a precise statement along such lines, it is necessary to first define a notion of *Dirichlet density* for a set of places of K.

Our aim in this handout is to define Dirichlet density in this broader context and to discuss the important Chebotarev density theorem that generalizes Dirichlet's theorem on primes in arithmetic progressions. The modern proofs of Chebotarev's theorem rest on class field theory, though historically Chebotarev proved his result without class field theory and his theorem played a pivotal role in motivating some aspects of Artin's work on class field theory.

2. Definition of Dirichlet density

Let K be a global field and let Σ be a set of non-archimedean places of K.

Definition 2.1. The *Dirichlet density* of Σ is

$$\delta_{\mathrm{Dir}}(\Sigma) = \lim_{s \to 1^+} \frac{\sum_{v \in \Sigma} q_v^{-s}}{\sum_v q_v^{-s}}$$

if this limit exists, where v ranges over non-archimedean places of K and q_v is the size of the residue field at v.

Of course, it should be explained why $\sum_v q_v^{-s}$ is finite for real $s > 1$. This sum is clearly bounded above by the product $\prod_{v \nmid \infty}(1 - q_v^{-s})^{-1} = \zeta_K(s)$ (as one sees by formally expanding out the geometric series expansions for the factors), and in class it was indicated how to prove the uniform and absolute convergence for $\zeta_K(s)$ with real $s > 1$ by giving upper bounded by constant multiples of $\zeta_{\mathbf{Q}}(s)$ or $\zeta_{\mathbf{F}_p(t)}(s)$, which in turn can be directly estimated "by hand". In this way we see that the fraction in the definition of Dirichlet density makes sense for $\mathrm{Re}(s) > 1$. The denominator in the definition of Dirichlet density can be replaced by a more concrete quantity, exactly as in the classical case $K = \mathbf{Q}$:

Lemma 2.2. *As $s \to 1^+$, $\sum_v q_v^{-s} \sim \log(1/(s-1))$.*

Proof. By Hecke and Tate (and Weil for global function fields), ζ_K has a meromorphic continuation to $\mathbf{C}$ with a simple pole at $s = 1$. Hence, for real $s > 1$ we have $\log \zeta_K(s) \sim \log(1/(s-1))$ as $s \to 1^+$. Thus, it suffices to prove

$$\sum_v q_v^{-s} \overset{?}{\sim} \log \zeta_K(s) = \sum_v -\log(1 - q_v^{-s})$$

as $s \to 1^+$. Since $-\log(1 - q_v^{-s}) = q_v^{-s} + \sum_{j \geq 2} 1/(j q_v^{sj})$ for $s > 1$ and we know $\log \zeta_K(s) \sim \log(1/(s-1))$ explodes as $s \to 1^+$, it suffices to show that the sum $\sum_{j \geq 2} \sum_v q_v^{-sj}$ is bounded for s near 1. In fact, we shall show that it is absolutely and uniformly convergent for $\mathrm{Re}(s) \geq 1/2 + \varepsilon$ for any $\varepsilon > 0$.

By expressing K as a finite separable extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$, and using that in a degree-d separable extension of global fields there are at most d places on the top field over a given place of the bottom field, with residue field degrees bounded by d as well, it suffices to treat the cases $K = \mathbf{Q}$ and $K = \mathbf{F}_p(t)$. Hence, it suffices to show that for real $s > 1/2$,

(1) the sum $\sum_{p, j \geq 2} p^{-js}$ is finite (this handles $K = \mathbf{Q}$),
(2) the sum $\sum_{r \geq 1, j \geq 2} p^r/(p^r)^{js}$ is finite for any prime p (this handles $K = \mathbf{F}_p(t)$, using the crude upper bound p^r on the number of places v of $\mathbf{F}_p(t)$ with $q_v = p^r$ for $r > 1$; there are $p + 1$ places with $q_v = p$, due to the infinite place.)

For the first of these two sums, we have

$$\sum_{p, j \geq 2} p^{-js} = \sum_p \sum_{j \geq 2} p^{-js} \leq \sum_p \frac{2}{p^{js}} \leq 2\zeta_{\mathbf{Q}}(2s),$$

giving the desired finiteness for $s > 1/2$. For the second of the sums of interest, we take $s = 1/2 + \varepsilon$ and get

$$\sum_{j \geq 2, r \geq 1} \frac{p^r}{(p^r)^{js}} = \sum_{j \geq 2, r \geq 1} \frac{1}{p^{rj(s - 1/j)}} \leq \sum_{j \geq 2, r \geq 1} \frac{1}{p^{rj\varepsilon}} = \sum_{j \geq 2} \left(\frac{1}{1 - p^{-\varepsilon j}} - 1 \right).$$

Letting $a = p^{-\varepsilon} \in (0, 1)$, we can rewrite this final sum as $\sum_{j \geq 2}((1 - a^j)^{-1} - 1)$, and so it suffices to prove finiteness of this latter sum for any $0 < a < 1$. Since $(1 - a^j)^{-1} - 1 = a^j/(1 - a^j)$ with $1/(1 - a^j)$ bounded above for $j \geq 2$, we get an upper bound by a constant multiple of $\sum_{j \geq 2} a^j$, and this is obviously finite. ∎

We conclude that an equivalent definition of Dirichlet density is

$$\delta_{\mathrm{Dir}}(\Sigma) = \lim_{s \to 1^+} \frac{\sum_{v \in \Sigma} q_v^{-s}}{\log(1/(s-1))}$$

when this limit exists. Since $\log(1/(s-1)) \to \infty$ as $s \to 1^+$, changing Σ by a finite set does not impact whether or not it has a Dirichlet density, nor the value when this density exists. Hence, for statements concerning Dirichlet density it is typical to be sloppy concerning ambiguities with finite sets of places (such as ramified places in a finite separable extension of K).

Obviously finite sets have Dirichlet density zero, and in general Dirichlet density lies in the interval $[0,1]$ when it exists. In particular, if a set Σ can be proved to have positive Dirichlet density, then it must be infinite. It is clear from the definition that a set Σ has a Dirichlet density if and only if its complement (in the set of non-archimedean places) has such a density, in which case the densities sum to 1. A subset of a set with Dirichlet density zero clearly has Dirichlet density that is moreover equal to 0, and a superset of a set with Dirichlet density 1 clearly has a Dirichlet density that is moreover equal to 1. It is easy to check that if $\Sigma_1, \dots, \Sigma_n$ are sets of non-archimedean places that admit Dirichlet densities $\delta_1, \dots, \delta_n$ and have overlaps $\Sigma_i \cap \Sigma_j$ with Dirichlet density 0 for $i \neq j$, then $\cup \Sigma_i$ has a Dirichlet density that is moreover equal to $\sum \delta_i$. However, Dirichlet density does not behave like a finitely additive measure. For example, if Σ and Σ' admit Dirichlet densities, it need not be the case that $\Sigma \cap \Sigma'$ or $\Sigma \cup \Sigma$ admit such densities (though clearly if one of them does then so does the other and the usual inclusion-exlcusion formula holds:

$$\delta(\Sigma_1 \cup \Sigma_2) = \delta(\Sigma_1) + \delta(\Sigma_2) - \delta(\Sigma_1 \cap \Sigma_2).)$$

Here is an interesting example of a set with full Dirichlet density:

Example 2.3. Let K'/K be a finite separable extension of global fields. The set Σ' of non-archimedean places v' on K' unramified over K and satisfying $f(v'|v) = 1$ has Dirichlet density 1 (where v is the place beneath v'); in the case that K'/K is Galois, these are precisely the places v' lying over the places v that are totally split in K'. Before we prove this, we warn that this density theorem for places of K' does *not* imply that the set of v in K that are totally split in K' has Dirichlet density 1. Indeed, as we shall see below, the Chebotarev density theorem will imply that if K'/K is Galois then the set of v in K that are totally split in K' has Dirichlet density $1/[K' : K]$. Thus, one should be careful to not confuse Dirichlet densities of sets on K with the sets over them on K'. Hence, it can be dangerous to view "Dirichlet density 1" as synonomous with "almost all places" when one moves between different global base fields.

To prove the density claim, we express K as a finite separable extension of a global field K_0 equal to $\mathbf{Q}$ or $\mathbf{F}_p(t)$, so K' is thereby realized as a finite separable extension of K_0. If v' on K' over v on K and over v_0 on K_0 satisfies $f(v'|v_0) = 1$ then obviously $f(v'|v) = 1$. Hence, it suffices to replace K with K_0, so we may assume $K = \mathbf{Q}$ or $K = \mathbf{F}_p(t)$. It is equivalent to show that the set Σ' of v' on K' with $f(v'|v) > 1$ has Dirichlet density 0. For such v' we have $q_{v'}^{-s} = q_v^{-f(v'|v)s} \leq q_v^{-2s}$ and there are at most $d = [K' : K]$ such v' over each v on K. Hence, the numerator $\sum_{v' \in \Sigma'} q_{v'}^{-s}$ in the definition of $\delta_{\mathrm{Dir}}(\Sigma')$ is bounded above by $[K' : K] \sum_v q_v^{-2s}$, and this is bounded for s near 1. Hence, dividing by $\log(1/(s-1))$ and sending $s \to 1^+$ gives a limit of 0.

There is an analogue of natural density for any global field K, as follows. For any $N > 0$, the set of non-archimedean places v on K that satisfy $q_v \leq N$ is a finite set. Indeed, by expressing K as a finite separable extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$ it suffices to treat these latter two cases, both of which are obvious. One important dichotomy between number fields and global function fields emerges, however: as N grows, the size of the set of v with $q_v \leq N$ grows with a linear bound for number fields but it has exponential growth for global function fields. Nonetheless, this finiteness result for each N permits us to define the *natural density* of a set Σ of non-archimedean places of K to be

$$\delta_{\mathrm{nat}}(\Sigma) = \lim_{x \to \infty} \frac{\#\{v \in \Sigma \,|\, q_v \leq N\}}{\#\{v \,|\, q_v \leq N\}}$$

if this limit exists.

3. Chebotarev's density theorem

For each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ and a positive prime $p \nmid N$, we have $\sigma = \mathrm{Frob}_p$ if and only if the isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \simeq (\mathbf{Z}/N\mathbf{Z})^\times$ carries σ to $p \bmod N$. Hence, an equivalent statement of Dirichlet's theorem on primes in arithmetic progressions is this:

Theorem 3.1 (Dirichlet)**.** *Choose $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ and let Σ be the set of prime ideals $v \nmid N$ of $\mathbf{Z}$ such that σ is the Frobenius element for v in $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$. The set Σ has a Dirichlet density, and it is equal to $1/[\mathbf{Q}(\zeta_N) : \mathbf{Q}]$.*

By using the formalism of Artin L-functions and class field theory, one can prove a vast generalization:

Theorem 3.2 (Chebotarev)**.** *Let K'/K be a finite Galois extension of global fields. Let c be a conjugacy class in $G = \mathrm{Gal}(K'/K)$. Let Σ_c be the set of non-archimedean places v of K that are unramified in K' and have Frobenius conjugacy class c. The set Σ_c has a Dirichlet density and it is equal to $|c|/|G|$. In particular, Σ_c is infinite.*

Corollary 3.3. *Let K'/K be a finite Galois extension of global fields. The set of v in K that are totally split in K' has Dirichlet density $1/[K' : K]$. In particular, it is infinite.*

Proof. This is the special case $c = \{1\}$. ■

We can generalize this corollary slightly, dropping the Galois condition:

Corollary 3.4. *Let K'/K be a finite separable extension of global fields. The set of v in K that are totally split in K' has a positive Dirichlet density equal to $1/[K'' : K]$ with K''/K a Galois closure of K'/K, and in particular it is an infinite set.*

Proof. Let K''/K' be a Galois closure of K'/K, so v is unramified in K'' if and only if it is unramified in K'. For such v, the condition that v be totally split in K' is that it be totally split in K'' (since K'' is a compositum of extensions of K that are abstractly isomorphic to K'). Hence, this set of v's has Dirichlet density $1/[K'' : K]$. ■

Let K be a global field and let S be a finite set of places that contains the archimedean places. Let $G_{K,S} = \mathrm{Gal}(K_S/K)$ be the Galois group for a maximal extension K_S/K unramified outside S; here, $K_S \subseteq K_{\mathrm{sep}}$ is the compositum of finite subextensions K'/K that are unramified outside S. For each $v \notin S$ and v' on K_S extending v, we get a well-defined Frobenius element $\phi(v'|v)$ in $G_{K,S}$. As we vary v', these sweep out a conjugacy class, called the *Frobenius conjugacy class* for v in $G_{K,S}$.

Theorem 3.5. *The set of Frobenius elements $\phi(v'|v) \in G_{K,S}$ is dense with respect to the Krull topology.*

In the language of modern algebraic geometry, $G_{K,S}$ is the étale fundamental group of the "punctured curve" $\mathrm{Spec}\, \mathscr{O}_{K,S}$ (with respect to the geometric generic point $\mathrm{Spec}\, K_{\mathrm{sep}}$ as base point). Hence, this theorem is an analogue of the obvious fact that the topological fundamental group of a finitely-punctured Riemann surface is generated by loops (which play the role of Frobenius elements).

Proof. By the definition of the Krull topology, it has to be proved that for each finite Galois subextension K'/K, every element of $\mathrm{Gal}(K'/K)$ is the image of $\phi(v'|v)$ for some v' on K_S over some $v \notin S$. By the functoriality of Frobenius with respect to passage to the quotient, for any v' on K_S over $v \notin S$, say with restriction w on K', the image of $\phi(v'|v)$ in $\mathrm{Gal}(K'/K)$ is $\phi(w|v)$. Hence, it is equivalent to prove that every element of $\mathrm{Gal}(K'/K)$ is a Frobenius element for a place over some $v \notin S$. The Chebotarev density theorem shows even more: every element of $\mathrm{Gal}(K'/K)$ is a Frobenius element relative to infinitely many places of K unramified in K'. ■

Example 3.6. Let $K/\mathbf{Q}$ be a finite Galois extension and let q be a prime of $\mathbf{Q}$ unramified in K. Choose an integer a not divisible by q, and choose $g \in \mathrm{Gal}(K/\mathbf{Q})$. We claim that there exist infinitely many positive primes p unramified in K such that $p \equiv a \bmod q$ and $g = \phi(\mathfrak{p}|p\mathbf{Z})$ for a prime $\mathfrak{p}$ of $\mathscr{O}_K$ over p.

The point is that since $\mathbf{Q}(\zeta_q)$ is totally ramified at q, it is linearly disjoint from K over $\mathbf{Q}$. Hence, the natural map $\mathbf{Q}(\zeta_q) \otimes_{\mathbf{Q}} K \to K(\zeta_q)$ is an isomorphism, and more specifically the natural map

$$\mathrm{Gal}(K(\zeta_q)/\mathbf{Q}) \to \mathrm{Gal}(K/\mathbf{Q}) \times \mathrm{Gal}(\mathbf{Q}(\zeta_q)/\mathbf{Q})$$

is an isomorphism. We may consider the ordered pair $(g, a \bmod q)$ on the right side as corresponding to an element γ in the Galois group on the left side, and the desired properties for p say exactly that it is unramified in $K(\zeta_q)$ and there exists a prime over p in $K(\zeta_q)$ whose Frobenius element in $\mathrm{Gal}(K(\zeta_q)/\mathbf{Q})$ is γ. Thus, applying Chebotarev's theorem to the Galois extension $K(\zeta_q)/\mathbf{Q}$ does the job.