Math 676. A lattice in the adele ring

Let K be a number field, and let S be a finite set of places of K containing the set S_∞ of archimedean places. The S-adele ring $\mathbf{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathscr{O}_v$ is an open subring of the adele ring $\mathbf{A}_K$, and it meets the additive subgroup K in the subring $\mathscr{O}_{K,S}$ of S-integers. In class we saw that K is discrete in $\mathbf{A}_K$ and $\mathscr{O}_{K,S}$ is discrete in $\mathbf{A}_{K,S}$. The purpose of this handout is to explain why each of these discrete subgroups is co-compact. The main input will be that $\mathscr{O}_{K,S}$ is a lattice in $\prod_{v \in S} K_v$, a fact that we saw in an earlier handout (and that was a key ingredient in the proof of the S-unit theorem).

1. The case of S-adeles

We first study $\mathbf{A}_{K,S}/\mathscr{O}_{K,S}$. This is a locally compact Hausdorff topological group since $\mathbf{A}_{K,S}$ is such a group and $\mathscr{O}_{K,S}$ is a closed subgroup. Using the continuous projection to the S-factor, we get a continuous surjection

$$\mathbf{A}_{K,S}/\mathscr{O}_{K,S} \twoheadrightarrow (\prod_{v \in S} K_v)/\mathscr{O}_{K,S}$$

to a compact group, and the kernel is surjected onto by the compact group $\prod_{v \notin S} \mathscr{O}_v$, so the kernel is itself compact. Hence, to infer compactness of $\mathbf{A}_{K,S}/\mathscr{O}_{K,S}$ we use the rather general:

Lemma 1.1. *Let G be a locally compact Hausdorff topological group, and let H be a closed subgroup. Give the coset space G/H its locally compact Hausdorff quotient topology. The group G is compact if and only if H and G/H are compact.*

Proof. If G is compact then its closed subset H is compact and its continuous image G/H is certainly compact. For the converse, choose a collection of open sets U_i in G with compact closure in G (such U_i's exist since G is locally compact and Hausdorff). The images U_iH/H in G/H are open since U_iH is open in G (as it is a union of translates U_ih for $h \in H$), and so by compactness of G/H finitely many of these cover G/H. That is, G is a union of finitely many U_iH's. Each U_i has compact closure C_i in G, so G is a union of finitely many C_iH's. Each C_iH is compact because it is the image of the compact product $C_i \times H$ under the continuous multiplication map for G, so G is a union of finitely many compact subsets. Hence, G is compact. ∎

2. The adelic case

Now we turn to the task of proving that $\mathbf{A}_K/K$ is compact. For any S, $\mathbf{A}_{K,S}$ is an open subring of $\mathbf{A}_K$, so the continuous map

$$\mathbf{A}_{K,S}/\mathscr{O}_{K,S} \to \mathbf{A}_K/K$$

modulo *discrete* subgroups is an open map. It is also injective (why?), and so it is an open embedding. Hence, $\mathbf{A}_K/K$ is the directed union of open subgroups $\mathbf{A}_{K,S}/\mathscr{O}_{K,S}$ that are *compact*. It follows that $\mathbf{A}_K/K$ is compact if and only if these open embeddings are equalities for S "large enough".

Let us try to understand the cokernel of the inclusion, namely $\mathbf{A}_K/(K + \mathbf{A}_{K,S})$. This is a discrete topological group (since $\mathbf{A}_{K,S}$ is an open additive subgroup of $\mathbf{A}_K$), and from the definitions we have

$$\mathbf{A}_K/\mathbf{A}_{K,S} = \bigoplus_{v \notin S} K_v/\mathscr{O}_v.$$

We have a natural diagonal embedding of K in here with kernel given by the set of elements of K that are v-adically integral for all $v \notin S$, which is to say the kernel of the diagonal map from K is $\mathscr{O}_{K,S}$. Hence, we have an inclusion

$$K/\mathscr{O}_{K,S} \hookrightarrow \bigoplus_{v \notin S} K_v/\mathscr{O}_v,$$

and our aim is to prove that this is an equality for S "large enough". Since $\mathscr{O}_{K,S}$ can always be arranged to have trivial class group by taking S to be large enough (for example, S consists of S_∞ and non-archimedean places corresponding to prime factors of a finite set of representatives for the elements of the class group of $\mathscr{O}_K$), it suffices to prove that this inclusion is an equality whenever $\mathscr{O}_{K,S}$ has trivial class group. Rather more generally:

Theorem 2.1. *Let A be an arbitrary Dedekind domain, with F its fraction field. The natural inclusion*

$$F/A \to \bigoplus_{\mathfrak{m}} F_{\mathfrak{m}}/A_{\mathfrak{m}}^{\wedge}$$

is an equality if A has trivial class group. Here, $\mathfrak{m}$ ranges over the maximal ideals of A, $F_{\mathfrak{m}}$ is the $\mathfrak{m}$-adic completion of F, and $A_{\mathfrak{m}}^{\wedge}$ is its valuation ring.

Taking $A = \mathscr{O}_{K,S}$ gives what we need, since the completions from the primes of $\mathscr{O}_{K,S}$ are exactly the K_v's and $\mathscr{O}_v$'s for $v \notin S$ (check!). Also, the map into the direct sum in the theorem makes sense because each $x \in F$ has image in $F_{\mathfrak{m}}$ that lies in $A_{\mathfrak{m}}^{\wedge}$ for all but finitely many $\mathfrak{m}$.

Before reading the proof of the theorem, it is instructive to first study the case $A = \mathbf{Z}$ by hand to get a feel for what is going on: can you show that $\mathbf{Q}/\mathbf{Z} \to \bigoplus_p \mathbf{Q}_p/\mathbf{Z}_p$ is an isomorphism "by hand"? (hint: $\mathbf{Q}/\mathbf{Z}$ is a torsion group, so it maps isomorphically to the direct sum of its p^{∞}-torsion subgroups.)

Proof. We first get rid of completions by checking that the natural map

$$F/A_{\mathfrak{m}} \to F_{\mathfrak{m}}/A_{\mathfrak{m}}^{\wedge}$$

is an isomorphism, where $A_{\mathfrak{m}}$ denotes the algebraic localization of A at $\mathfrak{m}$; this is the special case of the theorem when A is a discrete valuation ring. That is, for any discrete valuation ring R with fraction field L we claim that the natural map

$$L/R \to L^{\wedge}/R^{\wedge}$$

is an isomorphism, with $L^{\wedge}$ denoting the completion of L with respect to the canonical place arising from R and $R^{\wedge}$ denoting its valuation ring. The injectivity of this map is the old result $L \cap R^{\wedge} = R$ inside of $L^{\wedge}$, a fact we saw long ago. For surjectivity, we pick $\pi \in R$ that is a uniformizer and we choose a subset $\Sigma \subseteq R$ that is a set of representatives for the residue field of R, which in turn coincides with the residue field of the discrete valuation ring $R^{\wedge}$. By using π-adic expansions with coefficients in Σ, consideration of $L^{\wedge}/R^{\wedge}$ is tantamount to looking at just the "polar parts" of π-adic expansions in $L^{\wedge}$. Each such polar part is a *finite* sum $\sum_{i>0} \sigma_i \pi^{-i} \in L$, so the asserted surjectivity from L/R follows.

We may now restate our problem in purely algebraic terms: we wish to show that the natural inclusion

$$F/A \to \bigoplus_{\mathfrak{m}} F/A_{\mathfrak{m}}$$

is an isomorphism if A has trivial class group. Assume A has trivial class group. The target is generated by elements that are 0 away from a single factor, so it suffices to fix $\mathfrak{m}$ and a nonzero $\mathfrak{m}$-adic polar part $\xi \in F/A_{\mathfrak{m}}$, and we have to construct $x \in F$ that is integral away from $\mathfrak{m}$ but has $\mathfrak{m}$-adic polar part ξ. If ξ has $\mathfrak{m}$-adic order $-e$ with $e > 0$ and $\pi \in A$ is a principal generator of $\mathfrak{m}$, then there is a well-defined residue class

$$\pi^e \xi \in A_{\mathfrak{m}}/\mathfrak{m}^e A_{\mathfrak{m}} \simeq A/\mathfrak{m}^e$$

(the isomorphism goes from right to left, and is the crux of the matter). For any $x' \in A$ that represents this residue class in $A/\mathfrak{m}^e$, $x'/\pi^e \in F$ solves our problem: it is integral away from $\mathfrak{m}$ and has polar part ξ at $\mathfrak{m}$. ■

We conclude with a concrete example to illustrate the compactness of $\mathbf{A}_K/K$ in the special case $K = \mathbf{Q}$. We claim

$$\mathbf{A}_{\mathbf{Q}} = \mathbf{Q} + [0,1] \times \prod_p \mathbf{Z}_p,$$

which is to say that the compact subset $[0,1] \times \prod_p \mathbf{Z}_p \subseteq \mathbf{A}_{\mathbf{Q}}$ (with the induced topology given by its product topology!) maps continuously onto the quotient $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$, thereby "explaining" the compactness of the quotient in this case. For an arbitrary adele $x = (x_v) \in \mathbf{A}_{\mathbf{Q}}$, for each prime p we let $y_p = \sum_{i>0} a_i/p^i \in \mathbf{Q}$ (with $a_i \in \mathbf{Z}$, $0 \le a_i < p$) be the finite-tailed p-adic polar part of $x_p \in \mathbf{Q}_p$, so $y_p \in \mathbf{Z}[1/p]$ is integral away from p for each p and $y_p = 0$ for all but finitely many p (as $x_p \in \mathbf{Z}_p$ for all but finitely many p). Hence, if $\ell \neq p$ are distinct primes then y_ℓ has image in $\mathbf{Q}_p$ that lies in $\mathbf{Z}_p$, so the rational number $y = \sum_p y_p \in \mathbf{Q}$ makes sense and its image in $\mathbf{Q}_p/\mathbf{Z}_p$ is represented by y_p for each p. Thus, $x_p - y \in \mathbf{Q}_p$ lies in $\mathbf{Z}_p$ for every p

because modulo $\mathbf{Z}_p$ it is represented by $x_p - y_p \in \mathbf{Z}_p$ (due to the choice of y_p). To summarize, the diagonally embedded adele y in $\mathbf{A}_\mathbf{Q}$ differs from the adele x by an adele that is integral at *all* non-archimedean places. That is,

$$x - y = (x_v - y) \in \mathbf{R} \times \prod_p \mathbf{Z}_p \subseteq \mathbf{A}_\mathbf{Q}.$$

If we further replace y by a $\mathbf{Z}$-translate then we do not affect this property of $x - y$, but we can adjust the archimedean component of the difference to lie in $[0, 1]$. That is, for a suitable integer n we have $x - (y + n) \in [0, 1] \times \prod_p \mathbf{Z}_p$, giving the desired result. This argument shows slightly more: $[0, 1) \times \prod_p \mathbf{Z}_p$ maps bijectively onto $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$, so it serves as a "fundamental domain".

Remark 2.2. The preceding worked example applies quite generally to $\mathbf{A}_K$ whenever $\mathscr{O}_K$ has class number 1, and it says that in such cases if $P \subseteq \mathbf{R} \otimes_\mathbf{Q} K = \prod_{v|\infty} K_v$ is a fundamental parallelotope for the quotient $(\mathbf{R} \otimes_\mathbf{Q} K)/\mathscr{O}_K$ then $\mathbf{A}_K = K + P \times \prod_{v\nmid\infty} \mathscr{O}_v$, so $P \times \prod_{v\nmid\infty} \mathscr{O}_v$ serves as a "fundamental domain" for $\mathbf{A}_K/K$. Of course, in general one cannot expect to find such a simple description of a fundamental domain, and working directly with the adele ring and its quotients is the best way to bypass undue complications caused by class groups.