Algebraic Groups I. Homework 9

1. Read Appendix B in the book *Pseudo-reductive groups* to learn Tits’ structure theory for smooth connected unipotent groups over arbitrary fields $k$ with positive characteristic, and how $k$-tori act on such groups. Especially noteworthy are the results labelled B.1.13, B.2.7, B.3.4, and B.4.3.

2. Let $U$ be a smooth connected commutative affine $k$-group, and assume $U$ is $p$-torsion if char$(k) = p > 0$.
   (i) If char$(k) > 0$ and $U$ is $k$-split, use B.1.12 in *Pseudo-reductive groups* to prove $U$ is a vector group.
   (ii) Assume char$(k) = 0$. Prove that any short exact sequence $0 \to G_a \to G \to G_a \to 0$ is split. (Hint: log$(u)$ is an “algebraic” function on the unipotent points of Mat$_n$.) Deduce that $U \simeq G^N_a$, and prove that any action on $U$ by a $k$-split torus $T$ respects this linear structure.

3. Let $k'/k$ be a degree-$p$ purely inseparable extension of a field $k$ of characteristic $p > 0$.
   (i) Prove that $U = R_{k'/k}(G_m)/G_m$ is smooth and connected of dimension $p - 1$, and is $p$-torsion. Deduce it is unipotent.
   (ii) In the handout on quotient formalism, it is proved that any commutative extension of $G_a$ by $G_m$ over any field is uniquely split over that field. Prove that $R_{k'/k}(G_m)(k_a)[p] = 1$, and deduce that $U$ in (i) does not contain $G_a$ as a $k$-subgroup! (For a salvage, see Lemma B.1.10 in “Pseudo-reductive groups”: a $p$-torsion smooth connected commutative affine group over any field of characteristic $p > 0$ admits an étale isogeny onto a vector group.)

4. Let $G$ be a smooth group of finite type over a field $k$, and $N$ a commutative normal $k$-subgroup scheme.
   (i) Prove that the left $G$-action on $N$ via conjugation factors uniquely through an action of $G/N$ on $N$, and if $N$ is central in $G$ then prove that the action of $G$ on itself via conjugation uniquely factors through an action of $G/N$ on $G$. Describe this explicitly for $G = \text{SL}_n$ and $N = \mu_n$ over any field $k$, accounting for the fact that $\text{SL}_n(k) \to \text{PGL}_n(k)$ is generally not surjective.
   (ii) Prove the commutator map $G \times G \to G$ uniquely factors through a $k$-morphism $(G/Z_G) \times (G/Z_G) \to \mathcal{D}(G)$.

5. Let $B$ be a smooth connected solvable group over a field $k$.
   (i) If $B = G_m \rtimes G_a$ with the standard semi-direct product structure, prove that $Z_B(t, 0)$ is the left factor for any $t \in k^\times - \{1\}$.
   (ii) Deduce by inductive arguments resting on (i) that if $k = \bar{k}$ and $S \subset B(k)$ is a commutative subgroup of semisimple elements then $S \subset T(k)$ for some maximal torus $T \subset B$.
   (iii) Assume char$(k) \neq 2$ with $k = \bar{k}$, and let $G = \text{SO}_n$ with $n \geq 3$. Let $\mu \simeq \mu_n^{n-1}$ be the “diagonal” $k$-subgroup $\{(\zeta_i) \in \mu_n^n \mid \prod \zeta_i = 1\}$. Prove that the disconnected $\mu$ is maximal as a solvable smooth $k$-subgroup of $G$ and is not contained in any maximal $k$-torus of $G$ (hint: it has too much 2-torsion), so in particular is not contained in any Borel $k$-subgroup (by (ii))!

6. Let $G$ be a quasi-split smooth connected affine $k$-group, and $B \subset G$ a Borel $k$-subgroup. Let $T$ be a maximal $k$-torus in $B$.
   (i) Using conjugacy of maximal tori in $G_{\bar{k}}$, prove $g \mapsto gBg^{-1}$ is a bijection from $N_G(T)(\bar{k})/Z_G(T)(\bar{k})$ onto the set of Borel $\bar{k}$-subgroups containing $T_{\bar{k}}$. In particular, this set is finite.
   (ii) Using HW8 Exercise 4, prove that $N_G(T)(k_s)/Z_G(T)(k_s) \to N_G(\bar{k})/Z_G(\bar{k})$ is bijective, and deduce that every Borel subgroup of $G_{\bar{k}}$ containing $T_{\bar{k}}$ is defined over $k_s$.
   (iii) Assume that $T$ is $k$-split and $Z_G(T) = T$. Using Hilbert 90 and HW8 Exercise 4, prove that $N_G(T)(k)/T(k) \to N_G(T)(k)/Z_G(T)(k)$ is bijective. Deduce that every Borel subgroup of $G_{\bar{k}}$ containing $T_{\bar{k}}$ is defined over $k_s$! In each of the classical cases (GL$_n$, SL$_n$, PGL$_n$, Sp$_{2n}$, and SO$_n$), find all $\bar{B}$ containing the $k$-split maximal “diagonal” $T$. How many parabolic $k$-subgroups can you find containing one such $B$? (At least for GL$_n$, SL$_n$, and PGL$_n$, prove you have found all such parabolics.)
   (iv) Prove that each maximal smooth unipotent subgroup of $G_{\bar{k}}$ admits a conjugate contained in $B_{\bar{k}}$, and deduce that if $B \cap B' = T$ for another Borel $B'$ containing $T$ then $G$ is reductive. Use this with (iii) to prove reductivity for GL$_n$ ($n \geq 1$), SL$_n$ ($n \geq 2$), PGL$_n$ ($n \geq 2$), Sp$_{2n}$ ($n \geq 1$), and SO$_n$ ($n \geq 2$).