1. Let $A$ be a central simple algebra over a field $k$, $T$ a $k$-torus in $\mathbb{A}^\times$.

(i) Adapt Exercise 5 in HW5 to make an étale commutative $k$-subalgebra $A_T \subseteq A$ such that $(A_T)_k$ is generated by $T(k_s)$, and establish a bijection between the sets of maximal $k$-tori in $\mathbb{A}^\times$ and maximal étale commutative $k$-subalgebras of $A$. Deduce that $SL(A)$ is $k$-anisotropic if and only if $A$ is a division algebra.

(ii) For an étale commutative $k$-subalgebra $B \subseteq A$, prove $Z_A(B)$ is a semisimple $k$-algebra with center $C$.

(iv) If $T$ is maximal as a $k$-split subtorus of $\mathbb{A}^\times$ prove $T$ is the $k$-group of units in $A_T$ and that the (central) simple factors $B_i$ of $B_T := Z_A(A_T)$ are division algebras.

(v) Fix $A \simeq \text{End}_D(V)$ for a right module $V$ over a central division algebra $D$, so $V$ is a left $A$-module and $V = \prod V_i$ with nonzero left $B_i$-modules $V_i$. If $T$ is maximal as a $k$-torus in $\mathbb{A}^\times$, prove $V_i$ has rank 1 over $B_i$ and $D$, so $B_i \simeq D$. Using $D$-bases, deduce that all maximal $k$-split tori in $\mathbb{A}^\times$ are $\mathbb{A}^\times(k)$-conjugate.

2. For a torus $T$ over a local field $k$ (allow $R$, $C$), prove $T$ is $k$-anisotropic if and only if $T(k)$ is compact.

3. Let $Y$ be a smooth separated $k$-scheme locally of finite type, and $T$ a $k$-torus with a left action on $Y$. This exercise proves that $Y^T$ is smooth.

(i) Reduce to the case $k = \kbar$. Fix a finite local $k$-algebra $R$ with residue field $k$, and an ideal $J$ in $R$ with $Jm_R = 0$. Choose $y \in Y^T(R/J)$, and for $R$-algebras $A$ let $E(A)$ be the fiber of $Y(A) \to Y(A/JA)$ over $E(A) = 0$ and make it a $A_0$-module $F(A) := JA \otimes_k \text{Tan}_y(Y) = JA \otimes_{A_0} (A_0 \otimes_k \text{Tan}_y(Y))$ naturally in $A$ (denoted $v + y$).

(ii) Define an $A_0$-linear $T(A_0)$-action on $F(A)$ (hence a $T_R$-action on $F$), and prove that $E(A)$ is $T(A)$-stable in $Y(A)$ with $t_v + y = t_0.v + t.y$, for $v \in E(A)$, $t \in T(A)$, $v \in F(A)$, and $t_0 = t mod m_R$.

(iii) Choose $\xi \in E(R)$ and define a map of functors $h : T_R \to F$ by $T.R \to F$ by $T.\xi = h(t) + \xi$ for points $t$ of $T_R$: check it is a 1-cocycle, and is a 1-coboundary if and only if $E^{10}(R) \neq 0$. For $V_0 = J \otimes_k \text{Tan}_y(Y)$ use $h$ to define a 1-cocycle $h_0 : T \to V_0$, and prove $t_v + y = t_0.v + c_0(t), c$ is a $k$-linear representation of $T$ on $V_0 \otimes_k u$. Use a $T$-equivariant splitting (!) to prove $h_0$ (and then $h$) is a 1-coboundary; deduce $Y^T$ is smooth!

4. Let $G$ be a smooth $k$-group of finite type, and $T$ a $k$-torus equipped with a left action on $G$ (an interesting case being $T$ a $k$-subgroup acting by conjugation, in which case $G^T = Z_G(T)$).

(i) Use Exercise 3 to show $Z_G(T)$ is smooth, and by computing its tangent space at the identity prove for connected $G$ that $T \subset Z_G$ if and only if $T$ acts trivially on $\mathfrak{g} = \text{Lie}(G)$.

(ii) Assume $T$ is a $k$-subgroup of $G$ acting by conjugation. Using Exercise 4 of HW7 and the semisimplicity of the restriction to $T$ of $\text{Ad}_G : G \to GL(\mathfrak{g})$, prove $N_G(T)$ and $Z_G(T)$ have the same tangent space at the identity. Via (i), deduce that $Z_G(T)$ is an open subscheme of $N_G(T)$, so $N_G(T)$ is smooth and $N_G(T)/Z_G(T)$ is finite étale over $k$.

(iii) Assumptions as in (ii), the Weyl group $W = W(G,T)$ is $N_G(T)/Z_G(T)$. If $T$ is $k$-split, use the equality $\text{End}_k(T) = \text{End}_k(T_{k_s})$ to prove that $W(k) = W(k_s)$ and deduce that $W$ is a constant $k$-group. But show $N_G(T)(k)$ does not map onto $W(k)$ if $k$ is infinite and $K$ is a separable quadratic extension of $k$ such that $-1 \notin N_{K/k}(K^\times)$ (e.g., $k$ totally real and $K$ a CM extension, or $k = Q$ and $K = Q(\sqrt{3})$) with $G = SL(K) \simeq SL_2$ and $T$ the non-split maximal $k$-torus corresponding the norm-1 part of $K \subset \text{End}_k(K)$.

(iv) Prove that $N_G(T)(k) \to W(k) = W(\kbar)$ is surjective for the cases in HW6, Exercise 4(ii).

5. (i) For any field $k$, affine $k$-scheme $X$ of finite type, and nonzero finite $k$-algebra $k'$, define a natural map $j_{X,k'/k} : X \to \text{Res}_{k'/k}(X_{k'})$ by $(X(k') \to X(k' \otimes_k R)) = X_{k'}(k' \otimes_k R)$ for $k'$-algebras $R$. Prove $j_{X,k'/k}$ is a closed immersion and that its formation commutes with fiber products in $X$.

(ii) Let $G$ be an affine $k$-group of finite type. Prove that $j_{G,k'/k}$ is a $k$-homomorphism.

(iii) A vector group over $k$ is a $k$-group $G$ admitting an isomorphism $G \simeq G_a^n$, and a linear structure on $G$ is the resulting $G_a$-action. A linear homomorphism $G' \to G$ between vector groups equipped with linear structures is a $k$-homomorphism which respects the linear structures. For example, $(x,y) \mapsto (x, y + x^p)$ is a non-linear automorphism of $G_a^p$ (with its usual linear structure) when $char(k) = p > 0$.

For any $k$, prove $G_a$ admits a unique linear structure and its linear endomorphism ring is $k$. Giving $G_a^m$ and $G_a^n$ their usual linear structures, prove the linear $k$-homomorphisms $G_a^m \to G_a^n$ correspond to $\text{Mat}_{m \times n}(k)$. Are there non-linear homomorphisms if $char(k) = 0$?