Algebraic Groups I. Homework 7

0. (optional) Read the proof (Ch. III, §11 in Mumford's *Abelian Varieties*) of *Cartier's theorem*: group schemes G locally of finite type over a field of char. 0 are smooth! (This uses left-invariant derivations.)

1. (i) Prove that ∂_x is an invariant vector field on $\mathbf{G}_a$, and $t^{-1}\partial_t$ is an invariant vector field on $\mathbf{G}_m$.

(ii) Let A be a finite-dimensional associative k-algebra, and $\underline{A}^\times$ the associated k-group of units. Prove $\mathrm{Tan}_e(\underline{A}^\times) = A$ naturally, and that the Lie algebra structure is then $[a, a'] = aa' - a'a$. Using $A = \mathrm{End}(V)$, compute $\mathfrak{gl}(V)$. Use this to compute the Lie algebras $\mathfrak{sl}(V), \mathfrak{pgl}(V), \mathfrak{sp}(B), \mathfrak{gsp}(B), \mathfrak{so}(q)$.

(iii) Read Corollary A.7.6 and Lemma A.7.13 (and the paragraph preceding it) in "Pseudo-reductive groups". Compute the p-Lie algebra structure on $\mathrm{Lie}(\underline{A}^\times)$, $\mathrm{Lie}(\mathbf{G}_m)$, and $\mathrm{Lie}(\mathbf{G}_a)$ when $\mathrm{char}(k) = p > 0$.

2. Let G be a smooth group of dimension $d > 0$ over k.

(i) Define the concept of *left-invariant* differential i-form for $i \geq 0$, and prove the space $\Omega^{i,\ell}_G(G)$ of such form has dimension $\binom{d}{i}$. Compute the 1-dimensional $\Omega^{d,\ell}_G(G)$ for $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, and $\mathrm{PGL}(V)$.

(ii) Using right-translation, construct a linear representation of G on $\Omega^{d,\ell}_G(G)$; the associated character $\chi_G : G \to \mathbf{G}_m$ is the *modulus character*. Prove $\chi_G|_{Z_G} = 1$ and deduce that $\chi_G = 1$ if $G/Z_G = \mathscr{D}(G/Z_G)$.

(iii) (optional) If k is local (allow $\mathbf{R}$, $\mathbf{C}$) and X is smooth, use the k-analytic inverse function theorem to equip $X(k)$ with a functorial k-analytic manifold structure, and use k-analytic Change of Variables to assign a measure on $X(k)$ to a nowhere-vanishing $\omega \in \Omega^{\dim X}_X(X)$. (Serre's "Lie groups and Lie algebras" does k-analytic foundations.) Relate with Haar measures, and prove $\chi_G^{\pm 1}|_{G(k)}$ *is* the classical modulus character.

3. Let K/k be a degree-2 finite étale algebra (i.e., a separable quadratic field extension or $k \times k$), and let σ be the unique non-trivial k-automorphism of K; note that $K^\sigma = k$. A *σ-hermitian space* is a pair (V, h) consisting of a finite free K-module equipped with a perfect σ-semilinear form $h : V \times V \to K$ (i.e., $h(cv, v') = ch(v, v')$, $h(v, cv') = \sigma h(v, v')$, and $h(v', v) = \sigma(h(v, v'))$). Note $v \mapsto h(v, v)$ is a quadratic form $q_h : V \to k$ over k satisfying $q_h(cv) = \mathrm{N}_{K/k}(c) q_h(v)$ for $c \in K$, $v \in V$, and $\dim_k V$ is even ($\mathrm{char}(k) = 2$ ok!).

The *unitary group* $\mathrm{U}(h)$ over k is the subgroup of $\mathrm{R}_{K/k}(\mathrm{GL}(V))$ preserving h. Using $\mathrm{R}_{K/k}(\mathrm{SL}(V))$ gives the *special unitary group* $\mathrm{SU}(h)$. Example: $V = F$ finite étale over K with an involution σ' lifting σ, and $h(v, v') := \mathrm{Tr}_{F/K}(v\sigma'(v'))$; e.g., F and K CM fields, k totally real, and complex conjugations σ' and σ.

(i) If $K = k \times k$, prove $V \simeq V_0 \times V_0^\vee$ with $h((v, \ell), (v', \ell')) = (\ell'(v), \ell(v'))$ for a k-vector space V_0. Identify $\mathrm{U}(h)$ with $\mathrm{GL}(V_0)$ carrying $\mathrm{SU}(h)$ to $\mathrm{SL}(V_0)$. Compute q_h and prove non-degeneracy.

(ii) In the non-split case prove that $\mathrm{U}(h)_K \simeq \mathrm{GL}_n$ carrying $\mathrm{SU}(h)$ to SL_n ($n = \dim_K V$). Prove $\mathrm{U}(h)$ is smooth and connected with derived group $\mathrm{SU}(h)$ and center $\mathbf{G}_m$, and q_h is non-degenerate. Compute $\mathfrak{su}(h)$.

(iii) Identify $\mathrm{U}(h)$ with a k-subgroup of $\mathrm{SO}(q_h)$. Discuss the split case, and all cases with $k = \mathbf{R}$.

4. Let a smooth k-group H act on a separated k-scheme Y. For a k-scheme S, let $Y^H(S)$ be the set of $y \in Y(S)$ invariant by the H_S-action on Y_S (i.e., $y_{S'}$ is $H(S')$-invariant for all S-schemes S').

(i) If $k = k_s$, prove Y^H is represented by the closed subscheme $\cap_{h \in H(k)} Y^h$ where $Y^h = \alpha_h^{-1}(\Delta_{Y/k})$ for $\alpha_h : Y \to Y \times Y$ the map $y \mapsto (y, h.y)$. Then prove representability by a closed subscheme of Y for general k by Galois descent. Relate this to Exercise 2 in HW5.

(ii) For $y \in Y^H(k)$ explain why H acts on $\mathrm{Tan}_y(Y)$ and prove $\mathrm{Tan}_y(Y^H) = \mathrm{Tan}_y(Y)^H$.

(iii) Assume H is a closed subgroup of a k-group G of finite type, $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Prove $\mathrm{Tan}_e(Z_G(H)) = \mathfrak{g}^H$ via adjoint action. Also prove $\mathrm{Tan}_e(N_G(H)) = \cap_{h \in H(k)} (\mathrm{Ad}_G(h) - 1)^{-1}(\mathfrak{h})$ when $k = k_s$.

5. A diagram $1 \to G' \overset{j}{\to} G \overset{\pi}{\to} G'' \to 1$ of finite type k-groups is *exact* if π is faithfully flat and $G' = \ker \pi$.

(i) For any such diagram, prove $G'' = G/G'$ via π. Prove a diagram of k-tori $1 \to T' \to T \to T'' \to$ is exact if and only if $0 \to \mathrm{X}(T'') \to \mathrm{X}(T) \to \mathrm{X}(T') \to 0$ is exact (as $\mathbf{Z}$-modules).

(ii) If G' is finite then π is an *isogeny*. Prove that isogenies are *finite flat* with constant degree, and that $\pi_n : \mathrm{SL}_n \to \mathrm{PGL}_n$ is an isogeny of degree n. Compute $\mathrm{Lie}(\pi_n)$; when is it surjective?

(iii) Prove that a short exact sequence of finite type k-groups induces a left-exact sequence of Lie algebras, short exact if G and G' are smooth. (Smoothness of G can be dropped.)

(iv) Read §A.3 through Example A.3.4 in *Pseudo-reductive groups*, and prove $F_{X/k} : X \to X^{(p)}$ is finite flat of degree $p^{\dim X}$ for k-smooth X. Prove $\mathrm{Lie}(F_{G/k}) = 0$, and compute $F_{G/k}$ for $\mathrm{GL}(V)$ and $\mathrm{O}(q)$.