1. Use the method of proof of Proposition 4.10, Chapter I, to prove the following scheme-theoretic version: if $k$ is a field and a smooth unipotent affine $k$-group $G$ is equipped with a left action on a quasi-affine $k$-scheme $V$ of finite type then for any $v \in V(k)$ the smooth locally closed image of the orbit map $G \to V$ defined by $g \mapsto gv$ is actually closed in $V$.

(Hint: to begin, let $k[V]$ denote the $k$-algebra of global functions on $V$ and prove that $R \otimes_k k[V]$ is the $R$-algebra of global functions on $V_R$ for any $k$-algebra $R$. Use this to construct a functorial $k$-linear representation of $G$ on $k[V]$ respecting the $k$-algebra structure. Borel’s $K$ should be replaced with $k$ after passing to the case $k = \overline{k}$. Note that it is not necessary to assume Borel’s $F$ is non-empty; the argument directly proves $J$ meets $k^\times$, so $J = (1)$ and hence $F$ is empty.)

2. A $k$-homomorphism $f : G' \to G$ between $k$-groups of finite type is an isogeny if it is surjective and flat with finite kernel.

(i) Prove that a surjective homomorphism between smooth finite type $k$-groups of the same dimension is an isogeny. (The Miracle Flatness Theorem will be useful here.)

(ii) Prove that if a map $f : T' \to T$ between $k$-tori is an isogeny if and only if the corresponding map $X(T) \to X(T')$ between Galois lattices is injective with finite cokernel.

(iii) Prove the following are equivalent for a $k$-torus $T$: (a) it contains $G_m$ as a $k$-subgroup, (b) there exists a surjective $k$-homomorphism $T \to G_m$, and (c) $X(T)_\mathbb{Q}$ has a nonzero $\text{Gal}(k_s/k)$-invariant vector. Such $T$ is called $k$-isotropic; otherwise we say $T$ is $k$-anisotropic. In general, a smooth affine $k$-group is called $k$-isotropic if it contains $G_m$ as a $k$-subgroup, and $k$-anisotropic otherwise.

(iv) Let $T$ be a $k$-torus. Prove the existence of a $k$-split $k$-subtorus $T_s$ that contains all others, as well as a $k$-anisotropic $k$-subtorus $T_a$ that contains all others. Also prove that $T_s \times T_a \to T$ is an isogeny. Compute $T_s$ and $T_a$ for $T = R_{k'/k}(G_m)$ for a finite separable extension $k'/k$.

3. (i) For a $k$-torus $T$, prove the existence of an étale $k$-group $\text{Aut}_{T/k}$ representing the automorphism functor $S \rightsquigarrow \text{Aut}_S(T_S)$. (Hint: if $T$ is $k$-split then show that the constant $k$-group associated to $\text{Aut}(X(T)) \simeq \text{GL}_n(\mathbb{Z})$ does the job. In general let $k'/k$ be finite Galois such that $T_{k'}$ is $k'$-split, and use Galois descent.)

(ii) Using the existence of the étale $k$-group $\text{Aut}_{T/k}$, prove that if a connected $k$-group scheme $G$ is equipped with an action on $T$ then the action must be trivial. Deduce that if $T$ is a normal $k$-subgroup of a connected finite type $k$-group $G$ then it is a central $k$-subgroup. Give an example of a smooth connected $k$-group containing $G_a$ as a non-central normal $k$-subgroup. (Hint: look inside $\text{SL}_2$.)

4. Let $T$ be a $k$-torus in a $k$-group $G$ of finite type. This exercise uses $\text{Aut}_{T/k}$ from Exercise 3.

(i) Construct a $k$-morphism $N_G(T) \to \text{Aut}_{T/k}$ with kernel $Z_G(T)$. Prove $W(G, T) := N_G(T)/Z_G(T)$ is a natural finite subgroup of $\text{Aut}_G(X(T))$. If $f : G' \to G$ is surjective with finite kernel and $T'$ is a $k$-torus in $G'$ containing $\ker f$ with $f(T') = T$ then prove $W(G', T') \to W(G, T)$ is an isomorphism.

(ii) For $G = GL_n, PGL_n, SL_n, Sp_{2n}$, and $T$ the $k$-split diagonal maximal $k$-torus (so $Z_G(T) = T$), respectively identify $X(T)$ with $\mathbb{Z}^n$, $\mathbb{Z}^n/\text{diag}, \{m \in \mathbb{Z}^n \mid \sum m_j = 0\}$, and $\mathbb{Z}^n$. Prove $N_G(T)(k)/Z_G(T)(k) \subset \text{Aut}_Q(X(T)_\mathbb{Q})$ is $S_n$ for the first three, and $S_n \ltimes (-1)^n$ for $Sp_{2n}$, all with natural action. (Hint: to control $N_G(T)$, via $G \to \text{GL}(V)$ decompose $V$ as a direct sum of $T$-stable lines with distinct eigencharacters.)

5. Let $(V, q)$ be a non-degenerate quadratic space over a field $k$ with $\dim V \geq 2$. This exercise proves $\text{SO}(q)$ contains $G_m$ (i.e., it is $k$-isotropic in the sense of Exercise 2(iii)) if and only if $q = 0$ has a solution in $V - \{0\}$.

(i) If $q = 0$ has a nonzero solution $v$ in $V$, prove that $v$ lies in a hyperbolic plane $H$ with $H \cap H^\perp = V$. (If $\text{char}(k) = 2$ and $\dim V$ is odd, work over $\overline{k}$ to show $v \not\in V^\perp$.) Use this to construct a $G_m$ inside of $\text{SO}(q)$.

(ii) If $SO(q)$ contains $G_m$ as a $k$-subgroup $S$, prove that $q = 0$ has a nonzero solution in $V$. (Hint: apply Exercise 5(iii) in HW5 to the 2-dimensional $k$-split $k$-torus $T$ generated in $\text{GL}(V)$ by $S$ and the central $G_m$. If $A \simeq k^\times$ is the corresponding "$k$-split" commutative $k$-subalgebra of $\text{End}(V)$, prove the resulting inclusion $G_m = S \to T = R_{A/k}(G_m) = G_m$ is $t \mapsto (t^{a_1}, \ldots, t^{a_d})$. Use the $A$-module structure on $V$ to find a $k$-basis $\{e_i\}$ that identifies $S$ with $\text{diag}(t^{a_1}, \ldots, t^{a_d})$ for $n_1 \leq \cdots \leq n_d$ with $\sum n_i = 0$. Prove $n_i < 0 < n_d$, and if $q = \sum_{i,j} a_{ij} e_i x_j$ in these coordinates then prove $n_i + n_j = 0$ when $a_{ij} \neq 0$. Deduce $q(v) = 0$ for any $v$ in the span of the $e_i$ for which $n_i < 0$, or for which $n_i > 0$.)