1. Let $T \subset \text{Sp}_{2n}$ be the torus of points $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for diagonal $t \in \text{GL}_n$. Prove $Z_G(T) = T$ (so $T$ is a maximal torus!), and deduce $Z_{\text{Sp}_{2n}} = \mu_2$. See the handout on orthogonal groups for a computation of $Z_{\text{SO}(q)}$.

2. Prove that $\text{PGL}_n$ is smooth using the infinitesimal criterion, and prove that it is connected by a suitable “action” argument. Then read the handout on smoothness and connectedness for orthogonal groups.

3. Let $X$ be a scheme over a field $k$, and $x \in X(k)$. Recall that $\text{Tan}_x(X)$ is identified as a set with the fiber of $X(k[ε]) \to X(k)$ over $x$. Let $k[ε, ε'] = k[t, t']/(t, t')^2$, so this is 3-dimensional with basis $\{1, ε, ε'\}$.
   (i) For $c \in k$, consider the $k$-algebra endomorphism of $k[ε]$ defined by $ε \mapsto cc$. Show that the resulting endomorphism of $X(k[ε])$ over $X(k)$ restricts to scalar multiplication by $c$ on the fiber $\text{Tan}_x(X)$.
   (ii) Using the two natural quotient maps $k[ε, ε'] \to k[ε], k[ε, ε'] \to k[ε']$, define a natural map
   $$X(k[ε, ε']) \to X(k[ε]) \times_{X(k)} X(k[ε'])$$
   and prove it is bijective. Using the natural quotient map $k[ε, ε'] \to k[ε], k[ε, ε'] \to k[ε']$, show that the resulting map
   $$X(k[ε]) \times_{X(k)} X(k[ε']) \to X(k[ε])$$
   induces addition on $\text{Tan}_x(X)$: the $k$-linear structure on $\text{Tan}_x(X)$ is encoded by the functor of $X$!
   (iii) For $(X, x) = (G, e)$ with a $k$-group $G$, relate addition on $\text{Tan}_x(X)$ to the group law on $G$: for $m : G \times G \to G$, show that $\text{Tan}_e(G) \times \text{Tan}_e(G) = \text{Tan}_{(e, e)}(G \times G) \to \text{Tan}_e(G)$ is addition.

4. Let $A$ be a finite-dimensional associative algebra over a field $k$. Define the ring functor $A$ on $k$-algebras by $A(R) = A \otimes_k R$ and the group functor $A^\times$ by $A^\times(R) = (A \otimes_k R)^\times$.
   (i) Prove that $A$ is represented by an affine space over $k$. Using the $k$-scheme map $N_{A/k} : A \to A_k^1$ defined functorially by $u \mapsto \det(m_n)$, where $m_n : A \otimes_k R \to A \otimes_k R$ is left multiplication by $u \in A(R)$, prove that $A^\times$ is represented by the open affine subscheme $N_{A/k}^{-1}(G_m)$. (This is often called “$A^\times$ viewed as a $k$-group”, a phrase that is, strictly speaking, meaningless, since $A^\times$ does not encode the $k$-algebra $A$.)
   (ii) For $A = \text{Mat}_n(k)$ show that $A^\times = \text{GL}_n$, and for $k = \mathbb{Q}$ and $A = \mathbb{Q}(\sqrt{d})$ identify it with an explicit $\mathbb{Q}$-subgroup of $\text{GL}_2$ (depending on $d$).
   (iii) How does the kernel of $N_{A/k} : A^\times \to G_m$ (the group of norm-$1$ units) relate to Exercise 4(iii) in HW1 as a special case? For $A = \text{Mat}_n(k)$, show that this homomorphism is the $n$th power (!) of the determinant.

5. This exercise develops a very important special case of Exercise 4. Let $A$ be a finite-dimensional central simple algebra over $k$. By general theory, this is exactly the condition that $A_K \simeq \text{Mat}_n(K)$ as $k$-algebras (for some $n \geq 1$), and such an isomorphism is unique up to conjugation by a unit (Skolem-Noether theorem).
   (i) By a clever application of the Skolem-Noether theorem (see Exercise 30, Chapter 3 of the book by Farb/Dennis on non-commutative algebra), it is a classical fact that the linear derivations of a matrix algebra over a field are precisely the inner derivations (i.e., $x \mapsto yx - xy$ for some $y$). Combining this with length-induction on artin local rings, prove the Skolem-Noether theorem for $\text{Mat}_n(R)$ for any artin local ring $R$ (i.e., all $R$-algebra automorphisms are conjugation by a unit).
   (ii) Construct an affine $k$-scheme $I$ of finite type such that naturally $I(R) = \text{Isom}_R(A_R, \text{Mat}_n(R))$, the set of $R$-algebra isomorphisms. Note that $I(\overline{k})$ is non-empty! Prove $I$ is smooth by checking the infinitesimal criterion for $I_{\overline{k}}$ with the help of (i). Deduce that $A_K \simeq \text{Mat}_n(K)$ for a finite separable extension $K/k$.
   (iii) By (ii), we can choose a finite Galois extension $K/k$ and a $K$-algebra isomorphism $θ : A_K \simeq \text{Mat}_n(K)$, and by Skolem-Noether this is unique up to conjugation by a unit. Prove that for any choice of $θ$, the determinant map transfers to a multiplicative map $A_K^\times \to A_k^1$ which is independent of $θ$. Deduce that it is $\text{Gal}(K/k)$-equivariant, and so descends to a multiplicative map $\text{Nrd}_{A/k} : A \to A_k^1$ which “becomes” the determinant over any extension $F/k$ for which $A_F \simeq \text{Mat}_n(F)$. Prove that $\text{Nrd}_{A/k}^* = N_{A/k}$ (explaining the name reduced norm for $\text{Nrd}_{A/k}$), and conclude that $A^\times = \text{Nrd}_{A/k}^{-1}(G_m)$.
   (iv) Let $\text{SL}(A)$ denote the scheme-theoretic kernel of $\text{Nrd}_{A/k} : A^\times \to G_m$. Prove that its formation commutes with any extension of the ground field, and that it becomes isomorphic to $\text{SL}_n$ over $K$. In particular, $\text{SL}(A)$ is smooth and connected; it is a “twisted form” of $\text{SL}_n$. (This is false for $K_{A/k}$ whenever $\text{char}(k) | n$)