1. Let $G$ be a smooth connected affine group over a field $k$.
   
   (i) For a maximal $k$-torus $T$ in $G$ and a smooth connected $k$-subgroup $N$ in $G$ that is normalized by $T$, prove that $T \cap N$ is a maximal $k$-torus in $N$ (e.g., smooth and connected!). Show by example that $S \cap N$ can be disconnected for a non-maximal $k$-torus $S$. Hint: first analyze $Z_G(T) \cap N$ using $T \times N$ to reduce to the case when $T$ is central in $G$, and then pass to $G/T$.
   
   (ii) Let $H$ be a smooth connected normal $k$-subgroup of $G$, and $P$ a parabolic $k$-subgroup. If $k = \bar{k}$ then prove $(P \cap H)^0_{\text{red}}$ is a parabolic $k$-subgroup of $H$, and use Chevalley’s theorem on parabolics being their own normalizers on geometric points (applied to $H$) to prove $P \cap H$ is connected (hint: work over $\bar{k}$).
   
   (iii) Granting $Q = N_H(Q)$ scheme-theoretically for parabolic $Q$ in $H$ (Prop. 3.5.7 in Pseudo-reductive Groups; rests on structure theory of reductive groups), prove $P \cap H$ in (ii) is smooth. (Hint: prove $(P \cap H)^0_{\text{red}}$ is normal in $P$, hence in $P \cap H$!) In particular, $B \cap H$ is a Borel $k$-subgroup of $H$ for all Borels $B$ of $G$.

2. Let $k$ be a field, and $G \in \{\text{SL}_2, \text{PGL}_2\}$.
   
   (i) Define a unique $\text{PGL}_2$-action on $\text{SL}_2$ lifting conjugation. Prove a $k$-automorphism of $G$ preserving the standard Borel $k$-subgroup and the diagonal $k$-torus is induced by the action of a diagonal $k$-point of $\text{PGL}_2$.
   
   (ii) Prove that the homomorphism $\text{PGL}_2(k) \to \text{Aut}_k(G)$ is an isomorphism. In particular, every $k$-automorphism of $\text{PGL}_2$ is inner. Show that $\text{SL}_2$ admits non-inner $k$-automorphisms if and only if $k^x \neq (k^x)^2$.

3. Let $\lambda : G_m \to G$ be a 1-parameter $k$-subgroup of a smooth affine $k$-group $G$. Define $\mu : U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ to be multiplication. We seek to prove it is an open immersion. Let $g = \text{Lie}(G)$.
   
   (i) For $n \in Z$ define $g_n$ to be the $n$-weight space for $\lambda$ (i.e., $\text{ad}(\lambda(t))X = t^n X$). Define $g_{\lambda > 0} = \oplus_{n > 0} g_n$, and similarly for $g_{\lambda < 0}$. Prove $\text{Lie}(P_G(\lambda)) = g_{\lambda > 0}$, $\text{Lie}(U_G(\lambda)) = g_{\lambda < 0}$, and $\text{Tan}_{(e,e)}(\mu)$ is an isomorphism.
   
   (ii) If $\text{G = GL}(V)$ and the $G_m$-action on $V$ has weights $e_1 > \cdots > e_m$, justify the block-matrix descriptions of $U_G(\lambda^{\pm 1})$, $Z_G(\lambda)$, and $P_G(\lambda)$. Deduce $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth and have trivial intersection.
   
   (iii) Working over $\bar{k}$ and using suitable left and right translations by geometric points, prove that $d\mu(\xi)$ is an isomorphism for all $\bar{k}$-points $\xi$ of $U_G(\lambda^{-1}) \times P_G(\lambda)$. Deduce that if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth (OK for $\text{GL}(V)$ by (ii)) then $\mu$ induces an isomorphism between complete local rings at all $\bar{k}$-points, and conclude that $\mu$ is flat and quasi-finite. Hence, $\mu$ has open image in such cases.
   
   (iv) Using valuative criterion for properness, prove a flat quasi-finite separated map $f : X \to Y$ between noetherian schemes is proper if all fibers $X_y$ have the same rank. (Hint: base change to $\bar{Y}$ the spectrum of a dvr.) By Zariski’s Main Theorem, proper quasi-finite maps are finite. Deduce $\mu$ is an open immersion if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth with trivial intersection. (Hint: finite flat of fiber-degree 1 is isomorphism.)

   This settles GL($V$); handout on “dynamic approach to algebraic groups” yields the general case from this!

4. Let $\lambda : G_m \to G$ be a 1-parameter $k$-subgroup of a smooth affine $k$-group. For any integer $n \geq 1$, prove that $P_G(\lambda^n) = P_G(\lambda)$, $U_G(\lambda^n) = U_G(\lambda)$, and $Z_G(\lambda^n) = Z_G(\lambda)$.

5. Let $G$ be a reductive group over a field $k$, and $N$ a smooth closed normal $k$-subgroup. Prove $N$ is reductive. In particular, $\mathcal{P}(G)$ is reductive.

6. Prove that $\mu_n[d] = \mu_d$ for $d|n$, and that $\mathbb{Z}/n\mathbb{Z} \to \text{End}(\mu_n)$ is an isomorphism.

7. Prove that a rational homomorphism (defined in evident manner: respecting multiplication as rational map) between smooth connected groups over a field $k$ extends uniquely to a $k$-homomorphism. (Hint: pass to the case $k = k_s$ by Galois descent, and then use suitable $k$-point translations.)

8. (optional) Let $G$ be a smooth connected affine group over an algebraically closed field $k$, char($k$) = 0.
   
   (i) If all finite-dimensional linear representations of $G$ are completely reducible, then prove that $G$ is reductive. (Hint: use Lie-Kolchin, and behavior of semisimplicity under restriction to a normal subgroup. This will not use characteristic 0.)
   
   (ii) Conversely, assume that $G$ is reductive. The structure theory of reductive groups implies that $g$ is a semisimple Lie algebra, and a subspace of a finite-dimensional linear representation space for $G$ is $G$-stable if and only if it is $g$-stable under the induced action $g \to \text{End}(V)$ since char($k$) = 0. Prove that all finite-dimensional linear representations of $G$ are completely reducible.