

ALGEBRAIC GROUPS I. UNIPOTENT RADICALS AND REDUCTIVITY

In class, we have proved the important fact that over any field k , a non-solvable connected reductive group containing a 1-dimensional split maximal k -torus is k -isomorphic to SL_2 or PGL_2 . That proof relied on knowing that maximal tori remain maximal after a ground field extension to \bar{k} , and so relies on Grothendieck's theorem. But for algebraically closed fields there is no content to Grothendieck's theorem, so for $k = \bar{k}$ this rank-1 classification is simpler to prove.

The aim of this handout is to use the rank-1 classification (usually just over algebraically closed fields) to prove some important results on the behavior of unipotent radicals and the property of reductivity with respect to two ubiquitous operations on smooth connected affine groups over an arbitrary field k : the formation of quotient k -groups (modulo normal k -subgroup schemes) and the formation of centralizers of k -tori (which we have seen are always smooth and connected).

Recall that it was proved in class by elementary means that reductivity is inherited by smooth connected normal k -subgroups. More specifically, we proved that if $N \subseteq G$ is a smooth connected normal k -subgroup then $\mathcal{R}_u(N_{\bar{k}}) \subseteq \mathcal{R}_u(G_{\bar{k}})$ (so reductivity of G implies that of N). In fact, the inclusion $\mathcal{R}_u(N_{\bar{k}}) \subseteq N_{\bar{k}} \cap \mathcal{R}_u(G_{\bar{k}})$ of subgroup schemes of $G_{\bar{k}}$ (using scheme-theoretic intersection) is always an *equality*, but the proof rests on some non-trivial structural properties of reductive groups which have not yet been proved. (A proof is given in Proposition A.4.8 of “Pseudo-reductive groups”, working over \bar{k} there.) The main input is the non-obvious fact that the scheme-theoretic center of a connected reductive group is a subgroup scheme of a torus (see Corollary 2.2 below), and so has no nontrivial subgroup schemes which can arise as subgroup schemes of smooth unipotent groups (HW5, Exercise 1).

Notation. In what follows, G always denotes a smooth connected affine group over an arbitrary field k , unless we indicate otherwise. Also, following tradition, we often denote characters and cocharacters of tori in *additive* notation, for instance writing $-\lambda$ rather than λ^{-1} for the composition of a homomorphism $\lambda : \mathbf{G}_m \rightarrow T$ with inversion and likewise writing 0 to denote the trivial character of T . The reason for doing this is that it is convenient to work with the \mathbf{Q} -vector space $X(T)_{\mathbf{Q}}$ and to view the collections of characters and cocharacters as \mathbf{Z} -lattices.

1. PRELIMINARY RESULTS

Our first lemma will be surprisingly powerful (and is somewhat tricky to prove):

Lemma 1.1. *Assume $k = \bar{k}$, and let S be a k -torus in G . The Borel subgroups of $Z_G(S)$ are precisely the subgroups $Z_B(S) = B \cap Z_G(S)$ (scheme-theoretic intersection, as always) for Borel subgroups B of G which contain S .*

Proof. The smooth connected affine group $Z_G(S)$ contains a Borel subgroup B' , and S must lie in B' . Indeed, S lies in *some* Borel subgroup of $Z_G(S)$, all Borel subgroups in a smooth connected affine group over $k = \bar{k}$ are conjugate, and S is *central* in $Z_G(S)$, so indeed $S \subseteq B'$. In turn, B' is contained in a Borel subgroup B of G (via the characterization of Borel subgroups as maximal smooth connected solvable subgroups, rather than the “minimal parabolic” viewpoint). But $Z_B(S)$ is a smooth connected subgroup of B , so it is solvable, yet it lies in $Z_G(S)$. The inclusion $B' \subseteq Z_B(S) = B \cap Z_G(S)$ is therefore an equality by maximality of B' in $Z_G(S)$. Thus, we have found a Borel subgroup B in G containing S such that $Z_B(S)$ is equal to an arbitrarily chosen Borel subgroup B' of $Z_G(S)$. This proves that all Borel subgroups of $Z_G(S)$ have the asserted form.

Conversely, we wish to show that if B is a Borel subgroup of G containing S then the smooth connected solvable subgroup $Z_G(S) \cap B = Z_B(S)$ is a Borel subgroup of $Z_G(S)$. It suffices to prove that $Z_G(S)/Z_B(S)$ is complete. Since $S \subseteq B$, the S -conjugation on G preserves B and so induces an action on the complete coset space G/B . By HW8, Exercise 3, the scheme-theoretic fixed locus

$(G/B)^S$ is *smooth*. But this fixed locus is obviously closed in G/B , so it is complete. There is an evident map $f : Z_G(S)/Z_B(S) \rightarrow (G/B)^S$ which factors through the (irreducible) connected component of the identity of the target (since $Z_G(S)$ is connected), and we will show that it is an isomorphism onto this component. That will provide the desired completeness for $Z_G(S)/Z_B(S)$.

The map induced by f on tangent spaces at the identity is the natural map $\mathfrak{g}^S/\mathfrak{b}^S \rightarrow (\mathfrak{g}/\mathfrak{b})^S$ (see HW7, Exercise 4(ii)), and this is an isomorphism due to the complete reducibility of linear representations of tori (such as S). Hence, f is étale at the identity (since it is a map between *smooth* k -schemes, so the étale property near a k -point is equivalent to the isomorphism condition on tangent spaces there). But for any $g \in Z_G(S)(k) = Z_{G(k)}(S)$, f intertwines left multiplication by g on its source and target, so f is étale at all k -points and thus is étale. In particular, its image in the identity component of $(G/B)^S$ is open and f is flat. But f is clearly injective on geometric points, so the map f between its source and open image is flat with all fiber-ranks of degree exactly 1. Hence, by HW10, Exercise 3(iv), f is *finite* étale between its source and image, so finite flat of degree 1. In other words, f is an isomorphism onto its image, which is to say that f is an open immersion. The problem is to prove that f has image which fills up the entire identity component of $(G/B)^S$. We do this following the idea of the proof of Proposition 11.15 in Borel's book.

Consider the smooth surjective map $\pi : G \rightarrow G/B$ whose fibers are connected. The preimage $Y := \pi^{-1}((G/B)^S)$ is therefore connected and maps onto $(G/B)^S$. It suffices to prove that for all $y \in Y(k)$ there exists $b \in B(k)$ such that $yb^{-1} \in Z_G(S)(k)$. The definition of Y implies $sys^{-1} \in yB$ for all $s \in S(k)$, so $y^{-1}sy \in Bs = B$; i.e., $y^{-1}Sy \subseteq B$. That is, the action of the variety Y on G via $(y, g) \mapsto y^{-1}gy$ carries S into B . Thus, for the torus $T := B/\mathcal{R}_u(B)$ we get a map $Y \rightarrow \underline{\text{Hom}}(S, T)$ via

$$y \mapsto (s \mapsto y^{-1}sy \bmod \mathcal{R}_u(B)).$$

This map carries $1 \in Y$ to the natural map $j : S \rightarrow T$. Since $\underline{\text{Hom}}(S, T)$ is an étale k -scheme and Y is connected, it follows that Y is carried to the point $\{j\}$ ("rigidity of tori"). In other words, $y^{-1}sy \equiv s \bmod \mathcal{R}_u(B)$ for all s, y , so $y^{-1}Sy \subseteq S \rtimes \mathcal{R}_u(B)$.

The group $S \rtimes \mathcal{R}_u(B)$ is smooth and connected (even solvable), with S visibly a maximal torus, so all tori in this group of the same dimension as S are conjugate to S by a point of this group. Thus, $y^{-1}Sy = g^{-1}Sg$ for some $g \in S \rtimes \mathcal{R}_u(B)$. It is harmless to scale g on the left by points in S , so we can assume $g = b \in \mathcal{R}_u(B)$. Hence, $yb^{-1} \in N_G(S)$. But

$$y^{-1}sy \equiv s \equiv b^{-1}sb \bmod \mathcal{R}_u(B),$$

so the yb^{-1} -conjugation on S intertwines with the identity map on S via the natural map $j : S \rightarrow T := B/\mathcal{R}_u(B)$. Since $\ker j = S \cap \mathcal{R}_u(B) = 1$, it follows that $yb^{-1} \in Z_G(S)$, as desired. ■

Remark 1.2. The method of proof shows that if H is a (not necessarily normal or connected) smooth closed subgroup of G normalized by a torus S then $Z_G(S)/Z_H(S)$ is the identity component of $(G/H)^S$.

For a smooth connected affine group G over an algebraically closed field, since $\mathcal{R}_u(G)$ is normal and solvable in G it is contained in every Borel subgroup B of G . (Indeed, it is contained in some Borel subgroup, hence in all by conjugacy and normality arguments.) Hence, $\mathcal{R}_u(G)$ is contained in $\mathcal{R}_u(B)$ for every B , since such B are solvable and the unipotent radical is *functorial* for solvable smooth k -groups. The following result goes much deeper, and the proof will take a long time.

Theorem 1.3. *Let T be a maximal torus in a smooth connected affine group G over an algebraically closed field k . As B varies through the Borel subgroups which contain T , the resulting smooth*

connected unipotent subgroup

$$I(T) := \left(\bigcap_{B \supseteq T} \mathcal{R}_u(B) \right)_{\text{red}}^0$$

coincides with $\mathcal{R}_u(G)$. In particular, if G is reductive then $I(T) = 1$.

This result is quite striking, since a-priori it isn't evident that $I(T)$ is even normal in G , let alone equal to $\mathcal{R}_u(G)$. But there is a reason to expect this result: experience with many examples in the reductive case (for which the assertion is that $I(T) = 1$). In fact, once the structure theory of connected reductive groups is set up (in terms of root systems and root groups), it is easy to show that for any single Borel subgroup B containing a maximal torus T in a connected reductive group G , there is a unique B' containing T such that $\mathcal{R}_u(B) \cap \mathcal{R}_u(B') = 1$ scheme-theoretically (one calls B' the “opposite” Borel subgroup to B relative to T ; for $G = \text{GL}_n$ and the diagonal T and upper-triangular B , the lower-triangular Borel is B'). Thus, for a general smooth connected affine group G over $k = \bar{k}$, we may apply this to $G/\mathcal{R}_u(G)$ to get a pair of Borel subgroups B and B' containing T such that $\mathcal{R}_u(B) \cap \mathcal{R}_u(B') = \mathcal{R}_u(G)$ scheme-theoretically. This is a much stronger assertion than that $I(T) = \mathcal{R}_u(G)$, but it rests upon the finer structure theory of connected reductive groups not yet proved.

Proof. The torus T maps isomorphically onto a torus in $G/\mathcal{R}_u(G)$, and its image must be a maximal torus for dimension reasons (as the preimage in G of any torus in $G/\mathcal{R}_u(G)$ is clearly smooth connected and solvable). Thus, it is harmless to replace G with $G/\mathcal{R}_u(G)$ to reduce to the case when G is reductive. We aim to prove $I(T) = 1$.

If we can prove that $I(T)$ is *normal* in G then it must lie in $\mathcal{R}_u(G) = 1$, so we would be done. Such normality is not at all obvious, since $G(k)$ -conjugations move T all over the place! The crux of the matter is to prove that G is generated by some finite collection of smooth connected subgroups that each normalize $I(T)$ (so G does as well). We will achieve this by using the classification of connected reductive groups with a 1-dimensional maximal torus over algebraically closed fields: such groups are either SL_2 or PGL_2 , for which we can do some concrete calculations. (The intuition, for those familiar with the structure theory of complex semisimple Lie algebras, is that already for a single B and its “opposite” Borel with respect to T we should get a trivial intersection. The problem is that this intuition rests on the structure theory for such Lie algebras in terms of root systems, and the analogous structure theory for connected reductive groups rests on what we are presently trying to prove!)

Let $\Phi = \Phi(G, T)$ denote the set of *nontrivial* weights for the adjoint action of T on $\mathfrak{g} = \text{Lie}(G)$. We may (and do) assume Φ is non-empty. Indeed, otherwise $Z_G(T)$ has Lie algebra $\mathfrak{g}^T = \mathfrak{g}$ and thus $Z_G(T) = G$. But any smooth connected affine group over $k = \bar{k}$ with a central maximal torus must be solvable (the quotient by the maximal torus must be unipotent), and hence by reductivity we'd have $G = T$, leaving nothing to do.

For each $a \in \Phi$, $T_a := (\ker a)_{\text{red}}^0$ is a codimension-1 subtorus in T and $G_a := Z_G(T_a)$ is a smooth connected subgroup of G containing T with $\mathfrak{g}_a := \text{Lie}(G_a) = \mathfrak{g}^{T_a}$. In other words, \mathfrak{g}_a is the span of the weight spaces in \mathfrak{g} for those T -weights which kill T_a , or in other words are rational multiples of a in $X(T)_{\mathbf{Q}}$ (as $X(T/T_a)_{\mathbf{Q}}$ is 1-dimensional and contains $a \neq 0$). In particular, the trivial weight space $\mathfrak{g}^T = \text{Lie}(Z_G(T))$ is contained in every \mathfrak{g}_a , as is the a -weight space, so \mathfrak{g} is spanned by the \mathfrak{g}_a 's due to the complete reducibility of the T -action on \mathfrak{g} . Thus, G is generated by the subgroups G_a . It therefore suffices to prove that each G_a normalizes $I(T)$.

Note that by its definition, each G_a does contain $Z_G(T)$. In particular, T is a maximal torus in every G_a . We claim that each G_a is generated by its Borel subgroups that contain T . If G_a is

solvable (which is actually impossible, but we do not know that yet) then it is its own Borel subgroup and there is nothing to do. In the non-solvable case, passing to the non-solvable connected reductive quotient $G_a/\mathcal{R}_u(G_a)$ in which T maps isomorphically onto a maximal torus allows us to apply:

Lemma 1.4. *Let H be a non-solvable connected reductive group over an algebraically closed field, and assume H contains a maximal torus S such that all nontrivial S -weights occurring on \mathfrak{h} are \mathbf{Q} -multiples of each other.*

The quotient of H modulo its maximal central torus is either SL_2 or PGL_2 , there are exactly two Borel subgroups of H that contain S , and these Borel subgroups generate H .

Note that in the statement of the lemma we do not rule out *a priori* the possibility that the set of non-trivial S -weights on \mathfrak{h} is empty.

Proof. Consider the maximal smooth connected solvable normal subgroup R in H . This is reductive (since H is), so it is a torus. Being a normal torus in the connected H , it must be central. Thus, it is contained in S (as well as in every Borel) and is killed by all S -weights on \mathfrak{h} , so replacing H and S with H/R and S/R respectively is harmless. Thus, we may assume that there is no nontrivial central torus in H . We will next prove that $\dim S = 1$ (so we can apply the classification of non-solvable connected reductive groups with a 1-dimensional maximal torus!).

The set $\Phi(H, S)$ of nontrivial S -weights on \mathfrak{h} must be non-empty. Indeed, otherwise the Lie algebra \mathfrak{h}^S of the smooth connected subgroup $Z_H(S)$ fills up all of \mathfrak{h} , forcing $Z_H(S) = H$ for dimension and connectedness reasons. But then S is central, so $S = 1$. By maximality of S as a torus in H , it would then follow that H is unipotent, contradicting its non-solvability. By hypothesis, the elements of $\Phi(H, S)$ span a single line in $X(S)_{\mathbf{Q}}$, so for any $a \in \Phi(H, S)$ it follows that $S' := (\ker a)_{\mathrm{red}}^0$ is a codimension-1 torus in S on which all elements of $\Phi(H, S)$ act trivially. Hence, $Z_H(S') = H$ by the same Lie algebra considerations as just used, so $S' = 1$ since H has no nontrivial central torus. This proves $\dim S = 1$.

It follows from our classification of non-solvable connected reductive groups with a 1-dimensional maximal torus that necessarily H is isomorphic to either SL_2 or PGL_2 . By conjugacy of maximal tori, we can choose this isomorphism so that S goes over to the diagonal torus. The two standard Borel subgroups containing S in each case then generate H : for SL_2 we know that even their unipotent radicals do the job, and so the same holds for the quotient PGL_2 .

Finally, we prove that these two Borel subgroups are the only ones containing S . For any smooth connected affine group G over an algebraically closed field k and any maximal torus T in G , any two Borel subgroups B and B' in G that contain T are related through conjugation by an element in $N_G(T)$. (Indeed, $gBg^{-1} = B'$ for some $g \in G$, so gTg^{-1} and T are maximal torus in B' . Thus, for some $b' \in B'$ we have $b'gTg^{-1}b'^{-1} = T$, so $b'g \in N_G(T)$ does the job. Also see HW9 Exercise 6(i).) It follows that $N_G(T)/T$ acts transitively on the set of Borel subgroups containing T . For the groups SL_2 and PGL_2 , the diagonal torus has index 2 in its normalizer by inspection. (The case of PGL_2 can be reduced to SL_2 since the kernel of $\mathrm{SL}_2 \twoheadrightarrow \mathrm{PGL}_2$ is contained in the diagonal torus.) Hence, the two evident Borel subgroups containing the diagonal torus are the only ones. ■

Returning to our setup of interest, we conclude that G is generated by the Borel subgroups of the $G_a = Z_G(T_a)$ which contain T , so it suffices to prove that $I(T)$ is normalized by each such Borel subgroup. According to Lemma 1.1, the Borel subgroups of G_a are precisely $Z_B(T_a)$ for Borel subgroups B of G containing T_a , and such a subgroup contains T if and only if B does (as T obviously centralizes T_a !). Hence, G is generated by its subgroups $Z_B(T_a)$ as B varies through the Borel subgroups containing T . For such B , the smooth connected solvable group $Z_B(T_a)$ is $T \rtimes \mathcal{R}_u(B)^{T_a}$, so its unipotent radical is $\mathcal{R}_u(B)^{T_a}$.

If G_a is non-solvable then the maximal central torus in G_a is T_a (as this has codimension 1 in T and certainly T cannot be central as otherwise G_a/T would be unipotent, forcing G_a to be solvable). The visibly reductive quotient $G_a/(T_a \times \mathcal{R}_u(G_a))$ has T/T_a as a 1-dimensional maximal torus, so it is isomorphic to SL_2 or PGL_2 with T/T_a carried to the diagonal torus. In each of SL_2 and PGL_2 there are *exactly two* Borel subgroups containing the diagonal torus (by Lemma 1.4). Moreover, each such Borel subgroup supports (in the Lie algebra of its unipotent radical) exactly one of two nontrivial T -weights $\pm q_a \cdot a$ for some rational $q_a > 0$, *both* signs actually occur, and the corresponding weight spaces are 1-dimensional. Since T_a (and hence G_a) is insensitive to replacing a with a nonzero rational multiple (among the T -weights on \mathfrak{g}), it follows that each of $\pm q_a \cdot a$ is insensitive to replacing a with a positive rational multiple (among the T -weights on \mathfrak{g}).

If some G_a is equal to G then T_a is central in G and $\mathcal{R}_u(G_a) = 1$, so G/T_a is either SL_2 or PGL_2 , making it evident by inspection that G has exactly two Borel subgroups containing T and that their intersection is trivial. Hence, we may assume that all G_a are proper subgroups of G , so by induction on $\dim G$ each unipotent radical $\mathcal{R}_u(G_a)$ is the reduced identity component of the intersection of the $\mathcal{R}_u(B)^{T_a}$ for B containing T . Since torus centralizers are compatible with smoothness and with identity components (in the sense that they preserve connectedness), it follows that $\mathcal{R}_u(G_a) = I(T)^{T_a}$ for every a .

Under surjective homomorphisms between smooth connected affine groups over an algebraically closed field, Borel subgroups map onto Borel subgroups (proved in class) and hence likewise for their unipotent radicals (due to the structure of smooth connected solvable groups over $k = \bar{k}$). Thus, the image of each $\mathcal{R}_u(B)^{T_a}$ in $G_a/(T_a \times \mathcal{R}_u(G_a))$ is trivial when G_a is solvable (in which case $\mathcal{R}_u(B)^{T_a}$ clearly normalizes $I(T)$) and is one of two 1-dimensional possibilities when G_a is non-solvable. It follows that if G_a is non-solvable then $\mathcal{R}_u(B)^{T_a}$ contains $\mathcal{R}_u(G_a) = I(T)^{T_a}$ as a normal subgroup with codimension 1 and quotient whose Lie algebra supports a T -weight $\pm q_a \cdot a$ that is insensitive to replacing a with a positive rational multiple (among the T -weights on \mathfrak{g}). Moreover, this 1-dimensional quotient as a T -normalized subvariety of the coset space $G/I(T)^{T_a}$ depends *only* on the sign of the multiplier against a . Among *all* nonzero rational multiples of a which arise as T -weights on the tangent space at the identity for the coset space $G/I(T)$ it follows from Remark 1.2 (with $S = T_a$ and $H = I(T)$) that *exactly two* have weight space in \mathfrak{g} not entirely contained in $\mathrm{Lie}(I(T))$, and that these two weights are negatives of each other and have weight spaces meeting $\mathrm{Lie}(I(T))$ with codimension 1.

In what follows we only need to consider a such that G_a is non-solvable and (by replacing a with a uniquely determined positive rational multiple if necessary) the a -weight space is not entirely contained in $\mathrm{Lie}(I(T))$. The only other nonzero rational multiple of a which occurs in this way is $-a$ (and it does occur). Define $I_a(T)$ to be the identity component of the underlying reduced scheme of the intersection of those $\mathcal{R}_u(B)$ whose Lie algebra supports a as a weight outside of $\mathrm{Lie}(I(T))$. Clearly $I(T) \subseteq I_a(T)$, and each $\mathcal{R}_u(B)^{T_a}$ is contained in exactly one of $I_{\pm a}(T)$. Hence, if $I(T)$ is normal in each $I_{\pm a}(T)$ then it is normalized by every $\mathcal{R}_u(B)^{T_a}$, and we would be done. It therefore suffices to prove that each containment $I(T) \subseteq I_{\pm a}(T)$ between smooth connected unipotent subgroups is a normal subgroup. By renaming $-a$ as a if necessary (as we may do), we can focus on the containment $I(T) \subseteq I_a(T)$.

The preceding considerations yield the following very important consequence (especially after we finish the proof of Theorem 1.3, so $I(T) = 1$ in the reductive case):

Lemma 1.5. *The finite collection $\Psi(G, T) \subset X(T)$ of non-trivial T -weights on \mathfrak{g} whose weight spaces are not contained in $\mathrm{Lie}(I(T))$ is non-empty and stable under negation, with each such weight having a 1-dimensional weight space in the tangent space at the identity on the coset space $G/I(T)$. Moreover, for any $a \in \Psi(G, T)$, the set of \mathbf{Q} -multiples of a in $\Psi(G, T)$ is $\{\pm a\}$.*

Proof. For any such weight a , apply the preceding arguments and Remark 1.2 with $S = T_a$ and $H = I(T)$. ■

The normality of $I(T)$ in $I_a(T)$ is reduced to a dimension property, due to:

Lemma 1.6. *For any inclusion $U \hookrightarrow U'$ between smooth connected unipotent groups over a field, if $U \neq U'$ then $N_{U'}(U)$ is strictly larger than U . In particular, if $\dim(U'/U) = 1$ then U is normal in U' .*

Proof. We may assume the ground field is algebraically closed. The descending central series of U' (or consideration of upper-triangular unipotent matrices) forces U' to contain a central \mathbf{G}_a (here we use that the ground field is algebraically closed). If this is not contained in U then we win. Otherwise we can replace U and U' with their quotients modulo this common central subgroup and proceed by induction on $\dim U'$. ■

It now suffices to prove that $\dim I_a(T)/I(T) \leq 1$. The coset space $I_a(T)/I(T)$ has a natural T -action (as $I_a(T)$ and $I(T)$ are normalized by T), so its tangent space at the identity point is a direct sum of weight spaces for some T -weights; by the way we have chosen a , one such weight is a itself. The inclusion of $I_a(T)/I(T)$ into $G/I(T)$ implies that all nontrivial T -weights b occurring on $I_a(T)/I(T)$ have exactly a 1-dimensional weight space, and the only other nonzero rational multiple of b which can occur is $-b$, as we see by applying Remark 1.2 with $S = T_b$ and $H = I(T)$ and inspecting the Borel subgroups of SL_2 and PGL_2 containing the diagonal torus and having a specified nonzero weight on the Lie algebra of its 1-dimensional unipotent radical. Once again using Remark 1.2 (with the torus T_a), the weights 0 and $-a$ do *not* occur on $I_a(T)/I(T)$. Thus, we just have to rule out the occurrence of weights linearly independent from a .

Fix a choice of such a hypothetical extra weight b , so G_b is non-solvable (as otherwise $\mathcal{R}_u(B)^{T_b} = \mathcal{R}_u(G_b)^{T_b} = I(T)^{T_b} \subseteq I(T)$ for all Borel subgroups B of G containing T , contradicting that some $\mathcal{R}_u(B)$ has Lie algebra supporting the T -weight b outside of $\mathrm{Lie}(I(T))$). We will deduce a contradiction. Since b is linearly independent from a in $X(T)_{\mathbf{Q}}$, T_b and T_a are *distinct* codimension-1 subtori in T . By the choice of b , for every Borel subgroup B in G containing T such that $\mathrm{Lie}(\mathcal{R}_u(B))$ supports the T -weight a , this Lie algebra also supports the T -weight b outside of $\mathrm{Lie}(I(T))$. In particular, it *cannot* support the T -weight $-b$ (since $T_b = T_{-b}$ and we have analyzed $\mathcal{R}_u(B)^{T_b}$). In other words, if B is a Borel subgroup of G containing T such that $\mathcal{R}_u(B)^{T_a}$ has a as its unique weight modulo $I(T)$ then $\mathcal{R}_u(B)^{T_b}$ has b (and *not* $-b$) as its unique weight modulo $I(T)$. In particular, for such B we see that $\mathcal{R}_u(B)^{T_b}/I(T)^{T_b}$ is *uniquely determined* inside of $G_b/(T_b \cdot \mathcal{R}_u(G_b))$ ($= \mathrm{SL}_2$ or PGL_2). Hence, $\mathcal{R}_u(B)^{T_b}$ is also uniquely determined upon specifying that $\mathcal{R}_u(B)^{T_a}$ lifts the Borel subgroup B_a in $G_a/(T_a \cdot \mathcal{R}_u(G_a))$ whose Lie algebra supports the T/T_a -weight a (rather than $-a$), and therefore $B^{T_b} = T \rtimes \mathcal{R}_u(B)^{T_b}$ is uniquely determined upon specifying B^{T_a} lifts B_a .

Recall that T_a uniquely determines the pair $\{a, -a\}$. Call a codimension-1 torus S in T *singular* if there is a T -weight on \mathfrak{g} which kills S and whose weight space is not entirely contained in $\mathrm{Lie}(I(T))$. To get a contradiction, we will apply the following lemma.

Lemma 1.7. *If S and S' are distinct singular tori in T then there exist Borel subgroups B, B' in G containing T such that $B'^S = B^S$ and $B'^{S'} \neq B^{S'}$.*

Proof. We bring in the “dynamic approach” to algebraic groups (from an earlier handout, and discussion in class). Call a cocharacter $\lambda : \mathbf{G}_m \rightarrow T$ *regular* if it is not killed by any of the weights in $\Phi(G, T)$. This amounts to requiring that $\lambda \in X_*(T) = X(T)^\vee$ avoids finitely many “hyperplanes”, so there are many such λ . In particular, for all $a \in \Phi(G, T)$ the pairing $\langle a, \lambda \rangle = a \circ \lambda \in \mathrm{End}(\mathbf{G}_m) = \mathbf{Z}$ is nonzero. Attached to any regular λ (or even any 1-parameter subgroup of G at all), we obtained

smooth connected unipotent subgroups $U_G(\lambda)$ and $U_G(-\lambda)$, as well as a smooth connected subgroup $Z_G(\lambda) = Z_G(-\lambda)$, such that all are normalized by T and their Lie algebras are the respective weight spaces in \mathfrak{g} for the weights $a \in \Phi(G, T) \cup \{0\}$ satisfying $\langle a, \lambda \rangle > 0$, $\langle a, \lambda \rangle < 0$, and $\langle a, \lambda \rangle = 0$. The final case occurs precisely for $a = 0$ since λ is regular, so $Z_G(\lambda)$ and $Z_G(T)$ have the same Lie algebra and hence the containment $Z_G(T) \subseteq Z_G(\lambda)$ (which follows from the functorial characterization of $Z_G(\lambda)$ because λ is valued in T) is forced to be an equality due to connected and dimension reasons. Hence, we have an open immersion

$$U_G(\lambda) \times Z_G(T) \times U_G(-\lambda) \rightarrow G$$

via multiplication (see §1 of the handout “Dynamic approach to algebraic groups”, and HW10 Exercise 3), and $Z_G(T) = Z_G(\lambda) = Z_G(-\lambda)$ normalizes both $U_G(\lambda)$ and $U_G(-\lambda)$.

By Lemma 1.5, the nontrivial T -weights on \mathfrak{g} whose weight spaces are not entirely in $\text{Lie}(I(T))$ occur in opposite pairs $\pm a$, and no two such can arise in the Lie algebra of a *common* Borel subgroup B . Indeed, if they did then the image of $\mathcal{R}_u(B)^{T_a}$ in $G_a/(T_a \cdot \mathcal{R}_u(G_a))$ would be a Borel containing T/T_a and supporting a pair of opposite T/T_a -weights in its Lie algebra, an absurdity. Hence, if B is a Borel subgroup containing the smooth connected solvable subgroup

$$B(\lambda) := T \rtimes U_G(\lambda)$$

(so $B = T \rtimes U$ for $U = \mathcal{R}_u(B)$ containing $U_G(\lambda)$) then any T -weight on $U_G(-\lambda)$ occurring on $\text{Lie}(B)$ must have its *entire* weight space contained in $\text{Lie}(I(T))$. But $I(T) \subseteq B$ by definition of $I(T)$, so $\text{Lie}(I(T))$ maps onto the tangent space of $B/B(\lambda)$ at the identity. Thus, the multiplication map $B(\lambda) \times I(T) \rightarrow B$ is surjective on tangent spaces at the identity points, so $B = \langle B(\lambda), I(T) \rangle$.

Turning this argument around, we have shown that for any regular λ , $\langle B(\lambda), I(T) \rangle$ is a Borel subgroup of G containing T and the multiplication map $B(\lambda) \times I(T) \rightarrow \langle B(\lambda), I(T) \rangle$ is smooth at the identity point. Since $N_G(T)$ acts transitively on the set of Borel subgroups containing T (reviewed near the end of the proof of Lemma 1.4) and it visibly normalizes $I(T)$ and permutes the subgroups $B(\lambda)$, we conclude that the Borel subgroups of G containing T are *precisely* the subgroups $\langle B(\lambda), I(T) \rangle$, and that such a subgroup has Lie algebra spanned by $\text{Lie}(B(\lambda))$ and $\text{Lie}(I(T))$.

For any singular torus $S = T_a$ (with the label a chosen as above, uniquely up to sign), it follows that for $B = \langle B(\lambda), I(T) \rangle$ the Lie algebra of B^S is spanned by the Lie algebras of $B(\lambda)^S$ and $I(T)^S$ (since formation of torus centralizer commutes with the formation of Lie algebras for smooth connected affine groups). Hence, the Borel subgroup B^S in $G^S = G_a$ is generated by $B(\lambda)^S = T \rtimes U_G(\lambda)^S$ and $I(T)^S$, but $I(T)^S = I(T)^{T_a} = \mathcal{R}_u(G_a)$ is entirely determined by $T_a = S$: it has nothing to do with the choice of B ! Of course, the image of $U_G(\lambda)^S$ in $G^S/(S \times \mathcal{R}_u(G^S))$ is the unipotent radical of one of the two Borel subgroups containing T/S , and if we replace λ with $-\lambda$ then we get the “opposite” one (and not the trivial group, since $U_G(-\lambda)^S$ supports the entire $-a$ -weight space in \mathfrak{g} , which is *not* entirely contained in the Lie algebra of $\mathcal{R}_u(G^S) = I(T)^S$ due to the definition of “singular torus” and the occurrence in opposite pairs in Lemma 1.5).

Now we’re almost done. For the given pair of distinct singular tori $S = T_a$ and $S' = T_{a'}$ in T , pick $\lambda, \lambda' \in X_*(T) \subset X_*(T)_{\mathbf{Q}}$ such that

$$\langle a, \lambda \rangle, \langle a', \lambda \rangle > 0, \quad \langle a, \lambda' \rangle > 0 > \langle a', \lambda' \rangle.$$

Then the Borel subgroups $B = \langle B(\lambda), I(T) \rangle$ and $B' = \langle B(\lambda'), I(T) \rangle$ containing T satisfy $B^S = B'^S$ but $B^{S'} \neq B'^{S'}$.

Note also that if we replace λ' with $-\lambda'$ then we can also arrange that $\langle a', \lambda' \rangle > 0 > \langle a, \lambda' \rangle$. ■

Now we can complete the proof of Theorem 1.3. When constructing B and B' in the preceding lemma with $S = T_a = T_{-a}$ and $S' = T_b = T_{-b}$, we just need to exert control over which of the

two Borel subgroups of $G_a/(T_a \cdot \mathcal{R}_u(G_a))$ containing T/S is the image of $B'^S = B^S$: is it the one supporting the weight a or the weight $-a$. That is, our problem is not quite intrinsic to the distinct codimension-1 tori T_a and T_b , but involves the specific choices among the pairs $\pm a \in X(T/T_a)$ and $\pm b \in X(T/T_b)$. By introducing signs on λ and λ' if necessary, we can arrange our construction of B and B' to attain whatever signs we please. ■

Inspired by Theorem 1.3 let's now analyze the set of all Borel subgroups B containing a fixed maximal torus T in a smooth connected affine group G over an algebraically closed field k . The group $N_G(T)$ obviously acts on this collection, and we have:

Proposition 1.8. *The $N_G(T)$ -action by conjugation on the set of Borel subgroups containing T is transitive, and every such Borel subgroup contains $Z_G(T)$. The resulting transitive action of the finite group $W(G, T) = N_G(T)/Z_G(T)$ on the set of such Borel subgroups is simply transitive. In particular, the number of such Borel subgroups is finite, and in fact equal to $\#W(G, T)$.*

Proof. If B and B' are two Borel subgroups containing T then conjugacy of Borel subgroups gives $B' = gBg^{-1}$ for some $g \in G$, so T and gTg^{-1} are maximal tori in the smooth connected (solvable) B' . Hence, for some $b' \in B'$ we have $T = (b'g)T(b'g)^{-1}$, so the element $b'g \in N_G(T)$ conjugates B to B' . This proves the transitivity of the action.

Next we prove that $Z_G(T) \subseteq B$. Since $Z_G(T)$ is *normal* in $N_G(T)$, and $N_G(T)$ -conjugation transitively permutes the set of Borel subgroups containing T , it suffices to find *one* Borel subgroup B of G that contains $Z_G(T)$ (as then B also contains T and hence the $N_G(T)$ -conjugation takes care of the rest). We know that $Z_G(T)$ is smooth and connected, so to get containment in a Borel subgroup of G we just need to prove that it is solvable. Consider the quotient group $H = Z_G(T)/T$. This is a smooth connected affine group in which there are no nontrivial tori, due to the *maximality* of T , so it is necessarily unipotent (as we proved in class). Thus, H is solvable, so $Z_G(T)$ is indeed solvable.

For any $w \in W(G, T) = N_G(T)/Z_G(T)$ and $n \in N_G(T)$ representing w , the operation $B \mapsto nBn^{-1}$ on the set of Borel subgroups containing T only depends on $n \bmod Z_G(T) = w$ since $Z_G(T) \subseteq B$. Thus, we have a transitive action of $W(G, T)$ on the set of such Borel subgroups. It remains to prove that this is a simply transitive action, which is to say that if $n \in N_G(T)$ satisfies $nBn^{-1} = B$ for some $n \in N_G(T)$ then $n \in Z_G(T)$. In class we discussed the important theorem of Chevalley that every parabolic subgroup is its own normalizer (even in the scheme-theoretic sense, which we do not need); the proof was deferred to Borel's book, but the essential ingredients in that proof were covered in class. As a consequence of that result, $n \in B$, so $n \in N_B(T)$. Our problem is now intrinsic to B , or in other words we may rename B as G to reduce to the case when G is *solvable*. Then by maximality of T and the structure of solvable groups we can write $G = T \ltimes U$ for a smooth connected unipotent group U equipped with an action by T . Our goal is to prove that $N_G(T) = Z_G(T)$. The argument will be a trivial group theory calculation, not using *anything* about U beyond its smoothness!

It suffices to show that if $u \in U$ and $utu^{-1} \in T$ for all $t \in T$ then u is centralized by the T -action. It is harmless to multiply on the right by t^{-1} , so it is equivalent to say $u(tut^{-1}) \in T$ for all $t \in T$. But $tut^{-1} \in U$, so $u(tut^{-1}) \in U$. Thus, membership in T is equivalent to the condition $u(tut^{-1}) = 1$ which says exactly that u commutes with every $t \in T$; i.e., $u \in Z_G(T)$. ■

2. TORUS CENTRALIZERS AND UNIPOTENT RADICALS

The following theorem is the key miracle.

Theorem 2.1. *For any k -torus S in G , we have*

$$Z_G(S)_{\bar{k}} \cap \mathcal{R}_u(G_{\bar{k}}) = \mathcal{R}_u(Z_G(S)_{\bar{k}})$$

inside of $G_{\bar{k}}$. In particular, if G is reductive then so is $Z_G(S)$.

The preservation of reductivity under passage to torus centralizers in connected reductive groups is a powerful inductive technique to prove general theorems by dimension induction.

Proof. We may and do assume $k = \bar{k}$. The S -conjugation on G preserves the normal subgroup $\mathcal{R}_u(G)$, and the scheme-theoretic intersection $Z_G(S) \cap \mathcal{R}_u(G)$ is simply the S -centralizer $\mathcal{R}_u(G)^S$ in $\mathcal{R}_u(G)$ under this action. But functorial considerations make it clear that

$$Z_{S \ltimes \mathcal{R}_u(G)}(S) = S \times \mathcal{R}_u(G)^S,$$

and the left side is smooth and connected since it is a torus centralizer in the smooth connected affine group $S \ltimes \mathcal{R}_u(G)$! Thus, it follows that the direct factor (as a k -scheme) $\mathcal{R}_u(G)^S$ is also smooth and connected. (This same argument shows more generally that for *any* smooth connected subgroup H in G normalized by S , $Z_G(S) \cap H$ is smooth and connected.)

We conclude that $Z_G(S) \cap \mathcal{R}_u(G)$ is a smooth connected unipotent subgroup of $Z_G(S)$, and it is visibly normal (as $\mathcal{R}_u(G)$ is normal in G), whence $Z_G(S) \cap \mathcal{R}_u(G) \subseteq \mathcal{R}_u(Z_G(S))$. It remains to prove the reverse inclusion, which is to say that $\mathcal{R}_u(Z_G(S)) \subseteq \mathcal{R}_u(G)$. By the *functoriality* of unipotent radicals with respect to surjective homomorphisms between smooth connected affine groups (check!), it suffices to prove that under the quotient map $\pi : G \rightarrow G/\mathcal{R}_u(G)$, the image of $Z_G(S)$ is reductive (as then $\mathcal{R}_u(Z_G(S))$ must be killed in this image, and hence is killed by π). We know that the formation of torus centralizers commutes with the formation of images under homomorphisms between smooth connected affine groups (Corollary 1.3 in the handout “Dynamic approach ...”), so $\pi(Z_G(S))$ is the centralizer of the torus $\pi(S)$ in the reductive group $G/\mathcal{R}_u(G)$. Hence, we may rename $G/\mathcal{R}_u(G)$ as G to reduce to proving that if G is reductive then so is $Z_G(S)$.

The unipotent radical of any smooth connected affine group H (over $k = \bar{k}$) is smooth connected solvable and thus lies in *some* Borel subgroup. By conjugacy of Borel subgroups and normality of the unipotent radical, it follows that $\mathcal{R}_u(H)$ lies in *all* Borel subgroups of H , and thus (by solvability of Borel subgroups) in the *unipotent radicals* of all of these Borel subgroups. Taking $H = Z_G(S)$, we obtain from Lemma 1.1 that

$$\mathcal{R}_u(Z_G(S)) \subseteq \bigcap_{B \supseteq S} \mathcal{R}_u(Z_G(S) \cap B) \subseteq \bigcap_{B \supseteq S} \mathcal{R}_u(B)$$

since the formation of the unipotent radical is functorial in smooth connected solvable groups (such as with respect to the inclusion $Z_G(S) \cap B \rightarrow B$). Thus,

$$\mathcal{R}_u(Z_G(S)) \subseteq \left(\bigcap_{B \supseteq S} \mathcal{R}_u(B) \right)_{\text{red}}^0.$$

It therefore suffices to prove that this final intersection is trivial. If we pick a maximal torus T containing S , then the intersection can only grow if we restrict to those B that contain T . But restricting to such B yields the trivial group, by Theorem 1.3. ■

Corollary 2.2. *If G is a connected reductive group over a field k and T is a maximal k -torus then $Z_G(T) = T$; in particular, the scheme-theoretic center Z_G is contained in all such T .*

Also, for any surjective k -homomorphism $\pi : G \rightarrow G'$, $\pi(\mathcal{R}_u(G_{\bar{k}})) = \mathcal{R}_u(G'_{\bar{k}})$. In particular, if G is reductive then so is G' .

Our proof of the first assertion in this corollary will rest on Grothendieck's theorem concerning the existence of a maximal k -torus which remains maximal over \bar{k} , as that ensures $T_{\bar{k}}$ is maximal in $G_{\bar{k}}$. But we only apply the equality $Z_G(T) = T$ in the setup where $k = \bar{k}$ (e.g., in the proof of the behavior of unipotent radicals under quotient maps). Special cases were seen in HW3 Exercise 4(i) and HW4 Exercise 1.

Proof. We may and do assume $k = \bar{k}$. By Theorem 2.1, $Z_G(T)$ is reductive since G is reductive. But its maximal torus T is central, so the quotient $Z_G(T)/T$ is unipotent. Hence, $Z_G(T)$ is a solvable connected reductive group, so it is a torus (due to the structure of smooth connected solvable groups over algebraically closed fields). By maximality, the inclusion $T \hookrightarrow Z_G(T)$ must then be an equality.

Now consider the scheme-theoretic preimage of $\mathcal{R}_u(G')$ under the quotient map $G \rightarrow G'$. This is a normal subgroup scheme of G (since $\mathcal{R}_u(G')$ is normal in G'), so the identity component N of its underlying reduced scheme is as well. Then N inherits reductivity from G and admits $\mathcal{R}_u(G')$ as a quotient, so we can replace G with N to reduce to showing that for any connected reductive group G , a smooth connected unipotent quotient U of G must be trivial. Let T be a maximal torus in G . Its image in U is trivial, so by the compatibility of torus centralizers with respect to surjective homomorphisms between smooth connected affine groups (Corollary 1.3 in the handout "Dynamic approach ...") it follows that $U = Z_U(1)$ is the image of $Z_G(T) = T$. This forces $U = 1$ since U is unipotent and T is a torus. ■

Example 2.3. Consider a smooth affine group G over an algebraically closed field k , and any quotient $\pi : G \twoheadrightarrow G'$ with G' reductive. Thus, G^0 maps onto the reductive G'^0 , so by Corollary 2.2 the unipotent radical $\mathcal{R}_u(G) = \mathcal{R}_u(G^0)$ is killed by this quotient map. Hence, π factors uniquely through the natural quotient map $G \rightarrow G/\mathcal{R}_u(G)$, and conversely any quotient of G which factors through this latter map must be a quotient of $G/\mathcal{R}_u(G)$ and hence is reductive. For these reasons, the quotient $G/\mathcal{R}_u(G)$ is sometimes called the *maximal reductive quotient* of G .

Corollary 2.4. *Let G be a connected reductive group over a separably closed field k . Then G is generated by its maximal k -tori, and Z_G is the scheme-theoretic intersection of such tori.*

This corollary is actually true over any field, but the proof requires deeper structure theory (and extra care when k is finite).

Proof. Let N be the smooth connected k -subgroup generated by the maximal k -tori. Since $G(k)$ is Zariski-dense in G (as $k = k_s$) and it normalizes N , it follows that N is normal in G . Thus, G/N makes sense as a smooth connected group, and by construction it contains no nontrivial k -tori. By Grothendieck's theorem, such a group is unipotent. But G/N is reductive by Corollary 2.2, so it is trivial. Hence, $G = N$, so G is generated by its maximal k -tori T . It follows that Z_G is defined (functorially) by the condition of centralizing all such T . But $Z_G(T) = T$, so Z_G is the (scheme-theoretic) intersection of all such T . ■

We end this section with a surprisingly useful and non-obvious fact:

Corollary 2.5. *Let G be a connected reductive group over a field k of characteristic $p > 0$, and Z_G its scheme-theoretic center. Then $(Z_G)_{\bar{k}}$ cannot contain α_p or $\mathbf{Z}/p\mathbf{Z}$ as subgroup schemes. In particular, if U is a smooth unipotent k -subgroup of G then $Z_G \cap U = 1$ scheme-theoretically, and so $U \rightarrow G/Z_G$ is a closed k -subgroup inclusion.*

Beware that it *can* happen that a connected reductive group G contains a normal non-central infinitesimal subgroup scheme U having a composition series by α_p 's, though the resulting so-called

unipotent isogenies $G \rightarrow G/U$ only exist in characteristic 2. The simplest example is the weird purely inseparable isogeny $\mathrm{PGL}_2 \rightarrow \mathrm{SL}_2$ obtained by factoring the Frobenius isogeny $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2$ through the central quotient $\mathrm{SL}_2 \rightarrow \mathrm{PGL}_2$ whose kernel μ_2 is killed by Frobenius.

Proof. By Corollary 2.4, Z_G is a k -subgroup of a torus. Thus, we just have to check that \mathbf{G}_m does not contain α_p or $\mathbf{Z}/p\mathbf{Z}$. Since $\mathbf{G}_m[p] = \mu_p$, this amounts to the assertion that α_p and $\mathbf{Z}/p\mathbf{Z}$ are not isomorphic to μ_p . The case of $\mathbf{Z}/p\mathbf{Z}$ is clear since μ_p is not étale in characteristic p , and for α_p we can use the comparison of their p -Lie algebras to rule out an isomorphism (though Cartier duality provides another way, once one has computed that α_p is its own Cartier dual, which is not entirely trivial to verify directly.) ■

3. DERIVED GROUPS AND SEMISIMPLE GROUPS

Now we reap the fruit of our labors. A smooth connected affine group H over a field is called *perfect* if $H = \mathcal{D}(H)$. For example, if G is connected reductive over a field k then $\mathcal{D}(G)$ is a smooth connected normal k -subgroup of G , so it is also reductive. How about its own derived group $\mathcal{D}(\mathcal{D}(G))$? Can this decreasing chain involve several steps before it terminates (for dimension reasons), as happens for solvable groups? No, the process ends immediately:

Lemma 3.1. *Let G be a connected reductive group over a field k . The derived group $\mathcal{D}(G)$ is perfect.*

Proof. Let $N = \mathcal{D}(G)$. To prove that N is perfect, first note that $\mathcal{D}(N)$ is normal in G , so we may replace G with the (reductive!) quotient $G/\mathcal{D}(N)$ to reduce to the case when $\mathcal{D}(N) = 1$. In other words, the connected reductive group N is commutative, so it is a torus. But G/N is also commutative and reductive, hence a torus, so G must itself be a torus! But then obviously $N = 1$, which is perfect. ■

Recall that a solvable connected reductive k -group is just a k -torus by another name. By HW5 Exercise 4(iii), these are the same as the “Galois lattices” via the functor $T \rightsquigarrow \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$. The non-solvable case is much more interesting, and requires the apparatus of root systems to get a handle on the structure. The beginning of the story is:

Proposition 3.2. *Let G be a non-solvable connected reductive group over a field k , and assume it contains a split maximal k -torus T .*

- (1) *The set $\Phi(G, T)$ of non-trivial T -weights occurring on \mathfrak{g} is non-empty and stable under negation in $X(T)$, and for each $a \in \Phi(G, T)$ the weight space \mathfrak{g}_a is 1-dimensional and the only \mathbf{Q} -multiples of a in $\Phi(G, T)$ are $\pm a$.*
- (2) *For $a \in \Phi(G, T)$ and $G_a := Z_G(T_a)$, the natural map $T_a \times \mathcal{D}(G_a) \rightarrow G_a$ is a central isogeny with $\mathcal{D}(G_a)$ isomorphic to SL_2 or PGL_2 over k .*

Part (2) explains the importance of SL_2 and PGL_2 in the general theory of connected reductive groups (very similarly to the reason for the importance of \mathfrak{sl}_2 in the general theory of semisimple Lie algebras in characteristic 0).

Proof. By Theorem 1.3, $I(T) = 1$. Thus, we may apply Lemma 1.5 to get (1) (with the non-emptiness of $\Phi(G, T)$ being a consequence of the non-solvability of G , as we have seen earlier that if $\Phi(G, T)$ is empty then G is solvable). For (2), by Theorem 2.1 we know that the smooth connected k -subgroup G_a is reductive, so $\mathcal{R}_u(G_a) = 1$. Each G_a is *non-solvable*, because otherwise by reductivity such a G_a would be a torus, and hence equal to its maximal torus T , contradicting that $\mathrm{Lie}(G_a) = \mathfrak{g}^{T_a}$ contains the *non-zero* weight space in \mathfrak{g} for the nontrivial T -weight a . Thus, as

we saw in the proof of Theorem 1.3, when the quotient $G_a/T_a = G_a/(T_a \times \mathcal{D}_u(G_a))$ viewed over \bar{k} it is \bar{k} -isomorphic to SL_2 or PGL_2 , so its *split* k -torus T/T_a is maximal! Hence, by the classification result from class, this quotient group must be k -isomorphic to SL_2 or PGL_2 . These are their own derived groups by inspection (classical for SL_2 , hence inherited by the quotient PGL_2), so $\mathcal{D}(G_a) \rightarrow G_a/T_a$ is surjective.

It follows that $T_a \times \mathcal{D}(G_a) \rightarrow G_a$ is surjective, with central kernel given by the anti-diagonally embedded $T_a \cap \mathcal{D}(G_a)$. It therefore suffices to prove that the connected reductive $\mathcal{D}(G_a)$ has a 1-dimensional *split* maximal k -torus, as then it must be k -isomorphic to SL_2 or PGL_2 (by our classification theorem) and so by inspection has *finite* scheme-theoretic center (μ_2 and 1 respectively). That would force $T_a \cap \mathcal{D}(G_a)$ to be finite, completing the proof of (2).

For any smooth connected normal k -subgroup N in G the scheme-theoretic intersection $N \cap T$ is a maximal k -torus of N . Indeed, we may assume $k = \bar{k}$ (by Grothendieck's theorem!), and then for a maximal torus S in N and a maximal torus S' of G containing S we may find $g \in G(k)$ conjugating S' to T . Thus, it conjugates $S' \cap N$ to $T \cap N$, so $T \cap N$ contains the maximal torus gSg^{-1} of N . But then $T \cap N$ lies in the scheme-theoretic centralizer in N for the maximal torus gSg^{-1} , yet in any connected reductive group (such as N) every maximal torus is its own scheme-theoretic centralizer (Corollary 2.2). Hence, $T \cap N = gSg^{-1}$ is a maximal torus of N .

Now returning to our setup over a general field k , setting $N = \mathcal{D}(G_a)$ (normal in G_a) implies that $\mathcal{T} := T \cap \mathcal{D}(G_a)$ is a maximal k -torus in $\mathcal{D}(G_a)$. But this must be split (as T is split), so if the common dimension of the maximal k -tori of $\mathcal{D}(G_a)$ (split or not) is 1 then we will be done. (At this point we could increase k to be algebraically closed, but this is unnecessary so we do not.) It is a general fact that a central torus C in a smooth connected affine group H has *finite* intersection with $\mathcal{D}(H)$. (Proof: We can assume the ground field is algebraically closed, so C is split. Pick a faithful linear representation $H \hookrightarrow \mathrm{GL}(V)$, and form the weight decomposition $V = \bigoplus V_{\chi_i}$ with respect to the *faithful* C -action, so the χ_i generate $X(C)$ up to finite index. Then by centrality, H lands in $\prod \mathrm{GL}(V_{\chi_i})$, so $\mathcal{D}(H)$ projects to have determinant 1 in each factor. Thus, $C \cap \mathcal{D}(H)$ maps into each $\mathrm{GL}(V_{\chi_i})$ with scalar image killed by the determinant, hence inside the diagonal μ_{d_i} with $d_i = \dim V_{\chi_i}$. It follows that for $n = \prod d_i$ we have that χ_i^n kills $C \cap \mathcal{D}(H)$ for all i . But the χ_i generate a finite-index subgroup of $X(C)$, so $C \cap \mathcal{D}(H)$ is killed by a finite-index subgroup of $X(C)$. Hence, $C \cap \mathcal{D}(H)$ cannot contain any tori of positive dimension, so it is finite.) It follows that $T_a \cap \mathcal{D}(G_a)$ is finite, so $\mathcal{D}(G_a) \rightarrow G_a/T_a$ is an isogeny, with target isomorphic to SL_2 or PGL_2 . Thus, the split maximal torus \mathcal{T} in $\mathcal{D}(G_a)$ is 1-dimensional, so we are done. ■

Theorem 3.3. *Let G be a connected reductive group over a field k , and Z the maximal central torus. The formation of Z commutes with any extension of the ground field and the multiplication homomorphism*

$$Z \times \mathcal{D}(G) \rightarrow G$$

is an isogeny with central kernel.

In particular, $\mathcal{D}(G) \rightarrow G/Z$ and $Z \rightarrow G/\mathcal{D}(G)$ are isogenies with central kernel.

The basic example of this theorem is the central isogeny $\mathbf{G}_m \times \mathrm{SL}_n \rightarrow \mathrm{GL}_n$, whose kernel is the central anti-diagonal μ_n , along with the induced central isogenies $\mathrm{SL}_n \rightarrow \mathrm{PGL}_n$ and $\mathbf{G}_m \rightarrow \mathrm{GL}_n/\mathrm{SL}_n = \mathbf{G}_m$ given by $t \mapsto \det(\mathrm{diag}(t)) = t^n$.

Proof. The maximal central torus is the same as the maximal torus contained in the commutative scheme-theoretic center Z_G , so by consideration of n -torsion for n not divisible by $\mathrm{char}(k)$ we see that Z is the identity component of the Zariski closure of $\cup_n Z[n]$. Hence, the formation of Z commutes with extension of the ground field.

At the end of the proof of Proposition 3.2, we saw that the derived group of a smooth connected affine group always has finite intersection with any central torus. Hence, $Z \cap \mathcal{D}(G)$ is finite. It suffices to prove that G/Z is perfect. We claim that there are no nontrivial central tori in G/Z . If S is a central torus in G/Z then its preimage S' in G is a torus (being an extension of the torus S by the torus Z) and is also normal in G , forcing centrality in G (since G is connected). That implies $S' \subseteq Z$, so $S = 1$ as claimed.

We may now rename G/Z as G to reduce to showing that if $Z = 1$ then $\mathcal{D}(G) = G$. In particular, we may and do assume that k is algebraically closed (since we already proved that the formation of Z commutes with extension of the ground field). Pick a maximal torus T in G . By Corollary 3.2, for each $a \in \Phi(G, T)$, the a -weight space is 1-dimensional and the \mathbf{Q} -multiples of a in $\Phi(G, T)$ are precisely $\pm a$. The set $\Phi(G, T)$ generates a finite-index subgroup of $X(T)$. Indeed, otherwise there would be a nontrivial torus S in T killed by all elements of $\Phi(G, T)$, so $\mathfrak{g} = \mathfrak{g}^S = \text{Lie}(Z_G(S))$, forcing $Z_G(S) = G$ and so contradicting that we arranged for G to contain no nontrivial central tori.

For each $a \in \Phi(G, T)$, Proposition 3.2(2) ensures that for $T_a = (\ker a)_{\text{red}}^0$, the natural map

$$T_a \times \mathcal{D}(Z_G(T_a)) \rightarrow Z_G(T_a)$$

is a central isogeny. More specifically, $\mathcal{D}(Z_G(T_a))$ equipped with its T/T_a -action has exactly $\pm a$ as the nontrivial weights on its Lie algebra, and $S_a := T \cap \mathcal{D}(Z_G(T_a))$ is a 1-dimensional maximal torus of $\mathcal{D}(Z_G(T_a))$. Thus, the smooth connected subgroups $\mathcal{D}(Z_G(T_a))$ of $\mathcal{D}(G)$ generate a smooth connected subgroup H of $\mathcal{D}(G)$ whose Lie algebra supports all weight spaces for the nontrivial T -weights on \mathfrak{g} . Since \mathfrak{h} is a T -stable subspace of \mathfrak{g} which contains all weight spaces for nontrivial weights, whereas

$$\text{Lie}(T) = \text{Lie}(Z_G(T)) = \mathfrak{g}^T$$

is the weight space for the trivial weight, to prove that $G = \mathcal{D}(G)$ it remains to show that $T \subseteq \mathcal{D}(G)$.

We will prove that T is equal to the group $(N_G(T), T)$ generated by commutators $ntn^{-1}t^{-1}$ for $n \in N_G(T)(k)$ and $t \in T(k)$. Let $W = N_G(T)(k)/Z_G(T)(k) = N_G(T)(k)/T(k)$ denote the usual Weyl group which acts on T , so $(N_G(T), T)$ is the smooth connected subgroup of T generated by the images of the maps $T \rightarrow T$ defined by $t \mapsto (w.t)t^{-1}$ for $w \in W$. There is a natural action of W on the lattice $X_*(T)$ of cocharacters $\lambda : \mathbf{G}_m \rightarrow T$, and the sublattice $X_*((N_G(T), T))$ contains all elements $w.\lambda - \lambda$. Hence, to prove that the subtorus $(N_G(T), T)$ in T is full, it suffices to show that the elements $w.\lambda - \lambda$ generate a finite-index sublattice of $X_*(T)$, or equivalently that the $\mathbf{Q}[W]$ -module $X_*(T)_{\mathbf{Q}}$ has vanishing space of coinvariants. Since W is *finite* (HW8 Exercise 4(iii)), so $\mathbf{Q}[W]$ is semisimple, it is equivalent to have a vanishing space of W -invariants, which is to say that $X_*(T)^W = 0$. In other words, we claim that T^W is *finite*.

We will prove that $T^W \subseteq \ker(2a)$ (scheme-theoretically) for all $a \in \Phi(G, T)$, so $(T^W)_{\text{red}}^0 \subseteq (\ker(2a))_{\text{red}}^0 = (\ker a)_{\text{red}}^0 = T_a$. This is sufficient because the subtorus $(\cap_a T_a)_{\text{red}}^0$ in T is killed by all $a \in \Phi(G, T)$ and hence is trivial (as we have seen that $\Phi(G, T)$ generates $X(T)_{\mathbf{Q}}$, due to the arranged property $Z = 1$). Consider the group $G_a = Z_G(T_a)$ and its derived group $H_a = \mathcal{D}(G_a)$ (which is isomorphic to SL_2 or PGL_2), so T_a is the maximal *central* torus in G_a and $S_a := T \cap H_a$ is a maximal torus of H_a . Pick any representative $h_a \in H_a$ of the nontrivial element in $N_{H_a}(S_a)/S_a$, so h_a acts on S_a via inversion. It also centralizes T_a , and so normalizes $T_a \cdot S_a = T$. Thus, the class $w_a \in W$ of h_a centralizes T_a and swaps the weight spaces $\pm a$ for S_a , which in turn are the weight spaces for T/T_a acting on $\text{Lie}(G_a)$ (since $S_a \rightarrow T/T_a$ is an isogeny). In other words, the W -action on $X(T)$ negates $a \in X(T/T_a)$. It follows that if $t \in T^W(R)$ for a k -algebra R then

$$a(t) = a(w_a.t) = (w_a.a)(t) = (-a)(t),$$

so $(2a)(t) = 0$. In other words, $t \in \ker(2a)$, as desired. \blacksquare

Now we can finally prove the equivalence of several different ways to characterize *semisimple* groups:

Corollary 3.4. *Let G be a smooth affine group over a field k . The following conditions are equivalent.*

- (1) *The maximal smooth connected solvable normal subgroup $\mathcal{R}(G_{\bar{k}})$ of $G_{\bar{k}}$ is trivial.*
- (2) *The group G is reductive and has finite center.*
- (3) *The group G is reductive and G^0 is perfect.*

Condition (1) is the usual definition of semisimplicity, but sometimes one sees (2) or (3) used as cheap definitions. In practice it is important to know the equivalence among all of these conditions.

Proof. Under all hypotheses G is reductive and G^0 is readily checked to inherit all of the hypotheses from G , so we now may assume G is connected reductive. By Lemma 3.1 and Theorem 3.3, there is then a central isogeny

$$f : Z \times \mathcal{D}(G) \rightarrow G$$

where Z is the maximal central k -torus and $\mathcal{D}(G)$ is perfect. Thus, (2) implies (3). Likewise, if (3) holds then the isogeny property for f implies $Z = 1$, so there is no nontrivial central torus. But the scheme-theoretic center Z_G is contained in a maximal k -torus T , so $(Z_G)_{\text{red}}^0$ is a central torus and thus is trivial. This forces Z_G to be finite, so (2) holds. This proves the equivalence of (2) and (3).

It is clear that (1) implies $Z = 1$, and hence implies (3). Conversely, if (3) holds then $R = \mathcal{R}(G_{\bar{k}})$ is a smooth connected solvable normal subgroup of the perfect connected reductive group $G_{\bar{k}}$. Normality forces R to be reductive, and solvability forces it to be a torus. Normality in the connected $G_{\bar{k}}$ then forces this torus to be central. But (3) is equivalent to (2), so this central torus is trivial. Thus, (1) holds. \blacksquare

4. ROOT GROUPS AND ROOT DATA

Finally we come to the highlight of the basic theory: the link between connected reductive groups and combinatorial objects called *root data*. This link was first discovered in the theory of compact Lie groups and the structure theory of complex semisimple Lie algebras, where the slightly coarser notion of *root system* was used. Roughly speaking, root systems keep track of group-theoretic information “up to isogeny” whereas the root datum keeps track of information up to isomorphism. (The root datum viewpoint is also necessary for keeping track of the maximal central torus. But this was not regarded as an important piece of information in the early days of Lie groups, since a central torus is not particularly interesting from a representation-theoretic perspective.)

Throughout this section, G is a connected reductive group over a field k and T is a maximal k -torus that we assume to be k -split. We have seen in the homework that in many natural examples, there is no such T (e.g., unit groups of nontrivial central division algebras over k). Those G admitting such a T are called *k -split*. Note that since every maximal k -torus remains maximal after a ground field extension, and every torus splits over a finite Galois extension, loosely speaking every connected reductive k -group is a “twisted form” of a split one. Hence, the general nature of the classification of connected reductive groups comes in two parts: the combinatorial classification in terms of root data in the split case, which we will begin to discuss below, and then a Galois cohomological part to keep track of how “twisted” a given group is from a split one (thereby involving the structure of *automorphism groups* of split connected reductive groups, which is again best understood with the aid of root data, along with Galois cohomological methods).

Remark 4.1. Everything we do below will rest on the choice of T . Now of course it is typically not true (when $k \neq k_s$) that *every* maximal k -torus in k -split; already for GL_n this fails when k has degree- n finite separable extension fields. But it is true that all k -split T are $G(k)$ -conjugate. This is by no means obvious, and its proof rests on the structural understanding of the subgroup structure obtained via root data. Hence, one can keep in mind that at the end of the story all such choices of T will turn out to be “created equal”, and so in the end we will get results that are intrinsic to G up to $G(k)$ -conjugation (which is best possible, in some sense). For our purposes, the choice of T will simply be fixed throughout the discussion.

The beginning of our work is Proposition 3.2(1). The following terminology will be convenient:

Definition 4.2. The *roots* of the pair (G, T) are the non-trivial weights for T under its adjoint action on $\mathfrak{g} = \mathrm{Lie}(G)$. In other words, it is the set $\Phi(G, T) \subset X(T)$.

By Proposition 3.2(1), for each $a \in \Phi(G, T)$ the corresponding weight space \mathfrak{g}_a in \mathfrak{g} is 1-dimensional, and so we have a weight space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{a \in \Phi(G, T)} \mathfrak{g}_a \right)$$

with lines \mathfrak{g}_a , where $\mathfrak{t} = \mathrm{Lie}(T)$. In particular, $\Phi(G, T) = \emptyset$ if and only if $G = T$, which is to say that G is commutative (or equivalently, by reductivity, solvable). It is the non-solvable case which is the most important one, and we want to T -equivariantly “exponentiate” each \mathfrak{g}_a to a copy of \mathbf{G}_a in G . Ultimately this rests on a concrete calculation with SL_2 . First we prove the general result, and then we see what it says for SL_n .

Proposition 4.3. *For each root a of (G, T) , there is a unique smooth connected k -subgroup $U_a \subseteq G$ normalized by T such that the subspace $\mathrm{Lie}(U_a)$ equipped with its T -action is \mathfrak{g}_a . Moreover, $U_a \simeq \mathbf{G}_a$ as k -groups.*

The k -group U_a is called the *root group* in G attached to $a \in \Phi(G, T)$. Beware that it is crucial (in positive characteristic) to assume that U_a is T -normalized, not merely that its Lie algebra is T -stable under the adjoint action. Otherwise one can make counterexamples using the graph of Frobenius in $\mathbf{G}_a \times \mathbf{G}_a$.

Proof. Consider the unique codimension-1 torus $T_a = (\ker a)_{\mathrm{red}}^0$ in T killed by the nontrivial character a of T . The first task is to control all possibilities for U_a by proving that if $H \subseteq G$ is a T -normalized smooth connected k -subgroup for which $\mathrm{Lie}(H) = \mathfrak{g}_a$ then H is contained in the k -group $\mathcal{D}(Z_G(T_a))$ that we know to be k -isomorphic to SL_2 or PGL_2 . This is a geometric problem, so we may temporarily assume $k = \bar{k}$.

The Lie algebra condition forces H to be 1-dimensional, so H is either \mathbf{G}_a or GL_1 (since $k = \bar{k}$). The latter case is impossible, since then H would be a torus normalized by T , yet the T -action on H would then be trivial (since T is connected and $\underline{\mathrm{Aut}}(\mathrm{GL}_1) = \mathbf{Z}/2\mathbf{Z}$), contradicting the nontriviality of the T -action on $\mathrm{Lie}(H) = \mathfrak{g}_a$. Hence, H is unipotent.

Next we claim that the T_a -action on H must be trivial, so $H \subseteq G_a := Z_G(T_a)$. Since $H = \mathbf{G}_a$, for any $t \in T(k)$ the conjugation action of t on H is given by an algebraic group automorphism of \mathbf{G}_a , and the only such automorphisms are the nonzero constant scalings. In other words, t acts by some $\chi(t) \in k^\times$. But then the induced action on $\mathrm{Lie}(H) = \mathrm{Lie}(\mathbf{G}_a)$ is easily seen to also be scaling by the same $\chi(t)$ on this line, yet $\mathrm{Lie}(H) = \mathfrak{g}_a$ inside of \mathfrak{g} by hypothesis, so $\chi(t) = a(t)$. In particular, if $t \in T_a(k)$ then its action on H is trivial. Since H is unipotent and $G_a/\mathcal{D}(G_a)$ is a torus (see Proposition 3.2(2)) the containment of H in G_a forces $H \subseteq \mathcal{D}(G_a)$, as desired.

Now we return to the situation over a general field k , knowing that the only possibilities for U_a , if any is to exist at all, are to be found inside of the k -subgroup $\mathcal{D}(G_a)$ that we know to be

k -isomorphic to SL_2 or PGL_2 . In fact, the *proof* of existence of such a k -isomorphism arranged it so that *any* desired 1-dimensional k -split torus in $\mathcal{D}(G_a)$ is carried to the diagonal torus in SL_2 or PGL_2 . There is a natural such k -torus: $S_a := T \cap \mathcal{D}(G_a)$! Indeed, since $T_a \times \mathcal{D}(G_a) \rightarrow G_a$ is a central isogeny, and the scheme-theoretic preimage of T under this map is $T_a \times (T \cap \mathcal{D}(G_a))$, so to prove that $T \cap \mathcal{D}(G_a)$ is really a torus (then necessarily 1-dimensional and k -split due to the k -isogeny to the k -split T) we just have to prove the following auxiliary useful fact:

Lemma 4.4. *Let $f : H' \rightarrow H$ be a central isogeny between connected reductive groups over a field k . For every maximal k -torus T in H , the scheme-theoretic preimage $T' := f^{-1}(T)$ is a maximal k -torus in H' , and $T \mapsto T'$ is a bijection between the sets of maximal k -tori in H' and H .*

Proof. Once it is proved that T' is a torus, it must be maximal for dimension reasons (due to maximality of T in H), and the rest would then follow since the kernel is finite and we know that surjective homomorphisms carry maximal tori onto maximal tori. Thus, we may and do assume k is algebraically closed. For a maximal torus S' in H' , the image $f(S')$ is a maximal torus in H , so it has the form hTh^{-1} for some $h \in H$. Picking $h' \in f^{-1}(h)$, we may replace S' with $h'^{-1}S'h'$ to get to the case that $f(S') = T$. But $\ker f \subseteq Z_{H'} \subseteq Z_{H'}(S') = S'$, so $S' / (\ker f) = T$ inside of $H' / (\ker f) = H$. Hence, S' is the scheme-theoretic preimage of T , so we have reconstructed T' as a torus. ■

Returning to our situation of interest, we pick an isomorphism ϕ from $\mathcal{D}(G_a)$ onto SL_2 or PGL_2 such that S_a goes over to the diagonal torus D . Since $T = S_a \cdot T_a$ and T_a centralizes $\mathcal{D}(G_a)$, a k -subgroup of $\mathcal{D}(G_a)$ is T -normalized if and only if it is S_a -normalized, and then the action of T on its Lie algebra is uniquely determined by the action of S_a on the Lie algebra (as T_a must act trivially there). Hence, we have reduced everything to a very special case: G is either SL_2 or PGL_2 and T is the diagonal torus D ! This is so concrete that the rest will be a pleasant calculation.

By direct calculation with \mathfrak{sl}_2 and \mathfrak{pgl}_2 , the non-trivial weights for the adjoint D -action are easily seen (check!) to be the characters

$$a_+ : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^2, \quad a_- : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{-2}$$

in $X(D)$ in the SL_2 -case, and the characters

$$a_+ : \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto t, \quad a_- : \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto t^{-1}$$

in the PGL_2 -case, with respective weight spaces given respectively by the Lie algebras of the “upper triangular” unipotent subgroup U_+ and the “lower triangular” unipotent subgroup U_- . In both cases, by inspection we see that U_{\pm} are in fact normalized by D , and $U_{\pm} \simeq \mathbf{G}_a$ as k -groups. Thus, the existence part of the problem is settled, and it remains to prove *uniqueness*. In particular, now we may and do assume that $k = \bar{k}$, so any possibility which exists must be a copy of \mathbf{G}_a inside of our group.

Any possibility U for U_a yields a k -subgroup $D \ltimes U$ that is 2-dimensional, smooth, connected, and solvable, so by dimension reasons it must be a Borel subgroup that contains D . But we know from Proposition 1.8 that the set of Borel subgroups is permuted (simply) transitively by the group $W(G, D)$ of order 2, so the two such Borel subgroups $B_{\pm} = D \ltimes U_{\pm}$ are the *only* ones! This forces $U \subseteq B_{\pm}$, so $U = U_{\pm}$ for dimension reasons. Then correspondingly $\mathfrak{g}_a = \mathrm{Lie}(U) = \mathfrak{g}_{a_{\pm}}$, so the Lie algebra condition on U inside of \mathfrak{g} picks out exactly one of the two possibilities as the only one which can work, and we have seen that this possibility really does work. ■

Example 4.5. Let $G = \mathrm{SL}_n$ and $T = D$ the diagonal torus. Then for each $1 \leq i \neq j \leq n$ let U_{ij} be the k -subgroup $u_{ij} : \mathbf{G}_a \hookrightarrow G$ defined by setting $u_{ij}(x)$ to be the matrix whose diagonal entries are 1 and all other entries vanish except for the ij -entry which is x . This is easily seen to be a k -subgroup of G that is normalized by D , with $t = \mathrm{diag}(t_1, \dots, t_n)$ acting by $t \cdot u_{ij}(x) \cdot t^{-1} = u_{ij}((t_i/t_j)x)$. Thus, the space $\mathrm{Lie}(U_{ij}) \subset \mathfrak{sl}_n$ is a T -weight space for the nontrivial weight $a_{ij}(t) = t_i/t_j$. (Note that for $n = 2$ and $(i, j) = (1, 2)$, we get $t_1/t_2 = t_1^2$ since $t_2 = 1/t_1$ due to being in SL_2 .) This already gives us a collection of weight spaces filling up the entire dimension of \mathfrak{sl}_n away from the diagonal part \mathfrak{t} , so we have found all of the roots, as well as the root groups.

Another fun example is $G = \mathrm{Sp}_{2n}$ for a suitable “diagonal” T . This is worked out from scratch in the first few pages of §9.3 of “Pseudo-reductive groups”.

Having assembled the set of roots $\Phi(G, T)$ and the T -normalized root group $U_a \simeq \mathbf{G}_a$ inside of G for each root a , we next introduce the *coroots*. This will be a collection of nontrivial cocharacters $a^\vee : \mathrm{GL}_1 \rightarrow T$ which again arise from special arguments with SL_2 and PGL_2 :

Proposition 4.6. *For each $a \in \Phi(G, T)$, there is a unique k -homomorphism $a^\vee : \mathrm{GL}_1 \rightarrow S_a := T \cap \mathcal{D}(Z_G(T_a))$ such that $a \circ a^\vee \in \mathrm{End}(\mathrm{GL}_1) = \mathbf{Z}$ is 2; i.e., $a(a^\vee(t)) = t^2$. That is, relative to any k -isomorphism $u_a : \mathbf{G}_a \simeq U_a$, we have*

$$a^\vee(t)u_a(x)a^\vee(t)^{-1} = u_a(t^2x).$$

In the PGL_2 -case the map a^\vee is a degree-2 isogeny, and in the SL_2 -case it is an isomorphism.

Note that the choice of u_a really does not matter, since any two are related by composition with $\mathrm{Aut}_k(\mathbf{G}_a) = k^\times$, which clearly preserves the proposed condition.

Proof. The problem is intrinsic to the k -split pair $(\mathcal{D}(Z_G(T_a)), S_a)$ that we have seen is k -isomorphic to (SL_2, D) or (PGL_2, D) , and by composing such an isomorphism with a representative of the non-trivial class in the Weyl group of D if necessary we may arrange that the a -root group U_a goes over to the upper-triangular unipotent subgroup U_+ . So now the problem is an entirely concrete one about U_+ and D inside SL_2 and PGL_2 . In particular, we may and do use the choice $u_a(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. The existence of a^\vee is now by inspection: $a^\vee(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in the SL_2 -case and $a^\vee(t) = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}$ in the PGL_2 -case. For uniqueness it suffices to check on \bar{k} -points, and that is safely left to the reader. ■

Definition 4.7. The set of *coroots* of (G, T) is the subset $\Phi^\vee(G, T) \subset X_*(T)$ consisting of the cocharacters a^\vee for all $a \in \Phi(G, T)$.

By construction, $(-a)^\vee = -a^\vee$. We will see soon that a^\vee determines a , but if you think about it briefly this is not immediately obvious from the definitions. The first step towards understanding of coroots is to give an alternative way to think about them in terms of the finite *Weyl group* $W(G, T) = (N_G(T)/T)(k) = N_G(T)(k)/T(k)$ (latter equality by Hilbert 90, since T is k -split!). For each root a , the pair $(\mathcal{D}(G_a), S_a)$ is k -isomorphic to (SL_2, D) or (PGL_2, D) , and in particular has a Weyl group of order 2. All elements of $\mathcal{D}(G_a)$ centralize the codimension-1 torus T_a , so since $T = T_a \cdot S_a$ we see that any representative $n_a \in \mathcal{D}(G_a)$ of the non-trivial class in $W(\mathcal{D}(G_a), S_a)$ actually normalizes *all* of T and does not centralize it! That is, we have an *injective* homomorphism

$$W(\mathcal{D}(G_a), S_a) \hookrightarrow W(G, T).$$

We let $w_a \in W(G, T)$ denote the image of the nontrivial element in this order-2 subgroup. Under the natural faithful action of $W(G, T)$ on T , this element acts trivially on T_a and acts via inversion on S_a since it is represented by an element of $N_{\mathcal{D}(G_a)}(S_a)$ not centralizing S_a . Thus, on $X(T)_\mathbf{Q}$ it acts trivially on a hyperplane and via negation on a complementary line, so it is a *reflection*.

We define $s_a \in \mathrm{End}(X(T))$ to be the endomorphism induced by w_a .

Proposition 4.8. *Let $\langle \cdot, \cdot \rangle : X(T) \times X_*(T) \rightarrow \text{End}(\text{GL}_1) = \mathbf{Z}$ be the natural perfect pairing $\langle \chi, \lambda \rangle = \chi \circ \lambda$ between finite free \mathbf{Z} -modules. Then*

$$s_a(x) = x - \langle x, a^\vee \rangle a.$$

In particular, the Weyl element $w_a \in W(G, T)$ uniquely determines the coroot a^\vee .

There will be more work to do in order to show that a^\vee determines a .

Proof. By definition s_a is the action of w_a induced on $X(T)$. But w_a acts trivially on $T_a \subset \ker a$, and it acts by inversion on the subtorus S_a that is an isogeny complement to T_a . Thus, the isomorphism $X(T)_{\mathbf{Q}} \simeq X(S_a)_{\mathbf{Q}} \times X(T_a)_{\mathbf{Q}}$ induced by the isogeny $S_a \times T_a \rightarrow T$ implies that $s_a(a) = -a$ and s_a fixes a hyperplane pointwise, so it is a reflection on $X(T)_{\mathbf{Q}}$. Since it negates $a \neq 0$, necessarily $s_a(x) = x - \ell_a(x)a$ for a unique nonzero linear form ℓ_a on $X(T)_{\mathbf{Q}}$. Our problem is to prove that $\ell_a = \langle \cdot, a^\vee \rangle$.

Under the perfect duality pairing, the dual automorphism s_a^\vee on the dual lattice $X(T)^\vee = X_*(T)$ is also induced by the involution w_a acting on T (check!), so it fixes the hyperplane $X_*(T_a)_{\mathbf{Q}}$ pointwise and negates the line $X_*(S_a)_{\mathbf{Q}}$ through a^\vee . Hence, $s_a^\vee(\lambda) = \lambda - \langle x_a, \lambda \rangle a^\vee$ for a unique $x_a \in X(T)_{\mathbf{Q}}$. But it is easy to directly compute the dual of $x \mapsto x - \ell_a(x)a$, namely $\lambda \mapsto \lambda - \langle a, \lambda \rangle \ell_a$. This forces $\ell_a = a^\vee$ (and $x_a = a$). ■

Somewhat less evident is:

Proposition 4.9. *The surjective map of sets $\Phi(G, T) \rightarrow \Phi^\vee(G, T)$ defined by $a \mapsto a^\vee$ is bijective.*

Proof. Consider roots a and b such that $a^\vee = b^\vee$ in $X(T)$. Consider the element $w_a w_b \in W(G, T) \subset \text{GL}(X(T))$. This is the product $s_a s_b$, and from the explicit formulas

$$s_a(x) = x - \langle x, a^\vee \rangle a, \quad s_b(x) = x - \langle x, b^\vee \rangle b = x - \langle x, a^\vee \rangle b$$

it is easy to compute

$$s_a s_b(x) = x + \langle x, a^\vee \rangle (a - b).$$

Working in $X(T)_{\overline{\mathbf{Q}}}$, consider an eigenvector v of $s_a s_b$, so $s_a s_b(v) = cv$. Thus, $cv = v - \langle v, a^\vee \rangle (a - b)$. If $c \neq 1$ then v is a multiple of $a - b$, yet $a - b$ is fixed by $s_a s_b$ because

$$\langle a - b, a^\vee \rangle = \langle a, a^\vee \rangle - \langle b, a^\vee \rangle = \langle a, a^\vee \rangle - \langle b, b^\vee \rangle = 2 - 2 = 0.$$

This would force v to also be fixed by $s_a s_b$, contradicting that $c \neq 1$. In other words, $c = 1$ after all. That is, the only eigenvalue of $s_a s_b$ is 1, which is to say that $s_a s_b$ is unipotent. But $s_a s_b$ lies in the finite subgroup $W(G, T)$ on the automorphism group of $X(T)$, so unipotence forces this operator to be the identity.

We conclude that s_a and s_b are inverse to each other. Yet each is a reflection, hence of order 2, so in fact $s_a = s_b$. Now s_a is a reflection through the line spanned by a in $X(T)_{\mathbf{Q}}$, and likewise s_b is a reflection through the line spanned by b , so in fact $b \in \mathbf{Q} \cdot a$ in $X(T)_{\mathbf{Q}}$. By Proposition 3.2(1), this forces $b = \pm a$! Since $(-a)^\vee = -a^\vee \neq a^\vee$, the case $b = -a$ is ruled out. ■

We require one more elementary observation:

Proposition 4.10. *For each root a , the reflection $s_a : x \mapsto x - \langle x, a^\vee \rangle a$ on $X(T)$ preserves the finite set of roots $\Phi(G, T)$. Also the dual reflection*

$$s_a^\vee : \lambda \mapsto \langle a, \lambda \rangle a^\vee$$

on the dual lattice $X_(T)$ preserves the finite set of coroots $\Phi^\vee(G, T)$.*

Proof. By our preceding calculations, the actions of s_a and its dual are exactly the natural actions induced by the action of w_a on T . Thus, the first assertion is a consequence of the obvious fact that the action of $N_G(T)$ on T permutes the set $\Phi(G, T)$ of nontrivial T -weights on $\text{Lie}(G)$. For the second assertion, it is likewise suffices to prove that the $N_G(T)$ -action on T permutes the set of coroots. For any root a and any $n \in N_G(T)$ representing $w \in W(G, T)$, $w.a^\vee$ is a cocharacter of $nS_a n^{-1} = S_{w.a}$ (equality since $S_a := T \cap \mathcal{D}(Z_G(T_a))$ and $T_a := (\ker a)_{\text{red}}^0$). It is easy to check that it satisfies the property in Proposition 4.6 for the root $w.a$ (verify!), so it must be $(w.a)^\vee$. ■

Now we can finally state the definition we've been after:

Definition 4.11. A *root datum* is a 4-tuple (X, R, X^\vee, R^\vee) consisting of a pair of finite free \mathbf{Z} -modules X and X^\vee equipped with a perfect duality pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbf{Z}$ and a pair of finite subsets $R \subset X$ and $R^\vee \subset X^\vee$ such that there exists a bijection $a \mapsto a^\vee$ satisfying the following two axioms:

- (1) For all $a \in R$, $\langle a, a^\vee \rangle = 2$.
- (2) For all $a \in R$, the dual endomorphisms s_{a, a^\vee} of X and $s_{a^\vee, a}$ of X^\vee defined by

$$s_{a, a^\vee}(x) = x - \langle x, a^\vee \rangle a, \quad s_{a^\vee, a}(x^*) = x^* - \langle a, x^* \rangle a^\vee$$

satisfy $s_{a, a^\vee}(R) = R$ and $s_{a^\vee, a}(R^\vee) = R^\vee$.

Note that the first axiom forces $a, a^\vee \neq 0$, as well as the fact that s_{a, a^\vee} and $s_{a^\vee, a}$ respectively negate the lines through a and a^\vee and pointwise fix the hyperplanes orthogonal to a^\vee and a (working over \mathbf{Q} , say). Hence, each is a *reflection*.

There is a subtlety lurking here: we did not impose the specification of the bijection $a \mapsto a^\vee$ as part of the definition. Rather, this was simply assumed to exist in some way. Most textbooks impose the bijection as part of the structure of a root datum, and the entire basic theory can be developed in this way. But it is more elegant to not impose this, which we can do thanks to:

Proposition 4.12. *In a root datum, the bijection $a \mapsto a^\vee$ is uniquely determined. Writing $s_a := s_{a, a^\vee}$ and $s_{a^\vee} = s_{a^\vee, a} = s_{a^\vee}^\vee$, we also have $s_a(b)^\vee = s_{a^\vee}(b^\vee)$ for all $a, b \in R$.*

Proof. This is Lemma 3.2.4 in “Pseudo-reductive groups”; the proof is an elementary argument in linear algebra, relying on a small calculation via the axioms, given in SGA3. ■

The entire preceding analysis shows that to any split pair (G, T) we have associated a root datum

$$R(G, T) = (X(T), \Phi(G, T), X_*(T), \Phi^\vee(G, T)),$$

under which the reflections $s_a \in \text{End}(X(T))$ are induced by the elements $w_a \in W(G, T)$. Thus, the subgroup of $W(G, T)$ generated by the reflections s_a is intrinsic to the root datum, and it is denoted $W(R(G, T))$. The finiteness of this group is a general fact unrelated to algebraic groups:

Lemma 4.13. *The group $W(R) \subset \text{GL}(X)$ generated by the reflections s_a for $a \in R$ is finite.*

This is called the *Weyl group* of the root datum; it is trivial precisely when R is empty. The finiteness of $W(R)$ is proved in an elegant manner in Exercise 7.4.2 in Springer's book “Linear algebraic groups”. On the same page Springer reviews how a root datum with non-empty R gives rise to a *root system* (including the axioms for the latter), consisting of the nonzero \mathbf{Q} -vector space V spanned by R inside of $X_{\mathbf{Q}}$ and its finite set of non-zero vectors R that span it; the same exercise easily shows that the Weyl group of a root datum with non-empty R is naturally isomorphic to the Weyl group of the corresponding root system. In Bourbaki's LIE Chapter VI, it is root systems and not root data which are studied. This is akin to the dichotomy between isogeny classes of split connected semisimple groups and isomorphism classes of split connected reductive groups: all of

the real work is at the level of the root system, but the root datum is necessary to keep track of things at a level finer than isogenies.

The root data arising from split connected reductive groups have an extra property: if two roots are \mathbf{Z} -linearly dependent, they are the same up to a sign. Such root data are called *reduced*. The general case is not too far off from this:

Proposition 4.14. *If (X, R, X^\vee, R^\vee) is a root datum and $a, a' \in R$ satisfy $a = ca'$ for some $c \in \mathbf{Q}$ then $c \in \{\pm 1, \pm 2\}$.*

Proof. We may assume R is non-empty, and then this assertion is intrinsic to the associated root system. The result is then a basic fact proved early in the development of root systems; see Proposition 8(i) in §1.3 of Chapter VI of Bourbaki LIE. The argument is a nice bit of Euclidean geometry. ■

Remark 4.15. Just because split connected reductive groups only give rise to reduced root data, and so many texts ignored the non-reduced cases, the latter are important! First of all, in the study of connected reductive k -groups G which are not necessarily split but do contain a non-trivial k -split torus (perhaps not maximal as a k -torus), one associates a so-called *relative root datum* which is a root datum that can be non-reduced. These already show up in the classification of connected semisimple \mathbf{R} -groups which are not split and have non-compact group of \mathbf{R} -points. The same happens over all fields that aren't separably closed. They also show up in the theory of pseudo-reductive groups in characteristic 2.

In terms of the classification of root data via root systems, the only “irreducible” cases for which there are roots which are non-trivially divisible in the character lattice X are “simply connected” type C, which correspond to symplectic groups (so in fact non-reducedness is a somewhat “rare” occurrence, but it cannot be entirely ignored). For example, we saw by hand that SL_2 has its roots that are divisible by 2 in the character lattice; even for PGL_2 this does not happen.

The next step in the story is to formulate and prove the so-called Existence, Isomorphism, and Isogeny theorems which characterize isomorphism classes of k -split pairs (G, T) up to the $(T/Z_G)(k)$ -action on G in terms of root data, as well as characterize isogenies between two such pairs in terms of the root data. (Beware that typically $T(k)/Z_G(k)$ is smaller than $(T/Z_G)(k)$ when Z_G is not a torus, such as $G = \mathrm{SL}_n$ with $k^\times \neq (k^\times)^n$.) This can also be refined via an additional structure called a *pinning* which serves to get rid of the interference of the $(T/Z_G)(k)$ -action on G . These matters are explained in detail in Appendix A.4 of “Pseudo-reductive groups”, including the demonstration via faithfully flat descent that for the Isomorphism and Isogeny theorems it suffices to prove the results over algebraically closed fields, in which case there are multiple literature references one may consult. (For the Existence theorem one just has to find enough groups, and that was known classically away from the exceptional root systems.) In that Appendix A.4 the notion of *simply connected central cover* in the semisimple case is also discussed.

There is much more to say, such as relating the root datum to the subgroup structure. We end with just one observation along these lines, which is to prove (conditional on some general basic results in the theory of root systems) that the containment $W(R(G, T)) \subseteq W(G, T)$ is an equality (i.e., $W(G, T)$ is generated by the reflections s_a). This rests on:

Proposition 4.16. *A Borel subgroup B in G containing T is uniquely determined by the set $\Phi(B)$ of roots a such that $\mathfrak{g}_a \subseteq \mathrm{Lie}(B)$: explicitly, B is generated by T and the root groups U_a for all such a .*

Proof. By $N_G(T)$ -conjugacy among all such B , it suffices to prove the result for a single B . The theory of root systems provides the existence of linear forms on $X(T)_{\mathbf{Q}}$ that are non-vanishing on

the set of roots $\Phi(G, T)$ and meet each pair $\{\pm a\}$ in exactly one element. By fixing such a linear form and scaling it by a sufficiently divisible nonzero integer, we may arrange it to be \mathbf{Z} -valued, so it corresponds to a nontrivial k -homomorphism $\lambda : \mathrm{GL}_1 \rightarrow T$.

In terms of the handout “Dynamical approach . . .”, consider the corresponding smooth connected k -subgroup $B(\lambda) := P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ with $U_G(\lambda)$ a smooth connected unipotent k -subgroup whose Lie algebra is the span of the nonzero weight spaces for those $a \in X(T)$ such that $\langle a, \lambda \rangle > 0$. Such an a must be nonzero, and hence must be a root. But λ was rigged so that $\langle a, \lambda \rangle \neq 0$ for all roots a , so every root space \mathfrak{g}_a for a root a occurs inside the Lie algebra of either $U_G(\lambda)$ or $U_G(-\lambda)$. Likewise, the centralizer $Z_G(\lambda)$ of the subtorus $\lambda(\mathrm{GL}_1)$ is smooth and connected and contains T in its center, so its Lie algebra has no nonzero T -weights. Thus, the containment $T \subseteq Z_G(\lambda)$ is an equality on Lie algebras, and so is an equality of k -subgroups of G . It follows that $B(\lambda)$ is solvable.

Now we claim that $B(\lambda)$ is a Borel subgroup. Indeed, if not it would be contained in a Borel subgroup B' and for dimension reasons $\mathrm{Lie}(B')$ would then have to contain some weight space \mathfrak{g}_{-a} for a root a such that $\langle a, \lambda \rangle > 0$. In other words, $\pm a \in \Phi(B')$. So we just have to rule out such a possibility. Consider $B'_a := B' \cap Z_G(T_a)$. By Lemma 1.1, this is a Borel subgroup of $G_a := Z_G(T_a)$ that contains T . But G_a is generated by the central torus T_a and its derived group $\mathscr{D}(G_a)$, so for dimension reasons the containment $\mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a} \subseteq \mathrm{Lie}(G_a)$ is an equality. All of these weight spaces live in $\mathrm{Lie}(B'_a)$, so the containment $B'_a \subseteq G_a$ is an equality. In other words, G_a is solvable. That is absurd, since $\mathscr{D}(G_a)$ is visibly non-solvable (it is SL_2 or PGL_2)!

We conclude that $B(\lambda)$ is a Borel subgroup. For every root a whose weight space \mathfrak{g}_a lies in $\mathrm{Lie}(B(\lambda))$, consider the root group U_a . This is \mathbf{G}_a on which $\lambda(t)$ acts as scaling by $a(\lambda(t)) = t^{\langle a, \lambda \rangle}$ with $\langle a, \lambda \rangle > 0$. Thus, the functorial characterization of $U_G(\lambda)$ gives that $U_a \subseteq U_G(\lambda)$. Varying over all such a , the k -subgroups U_a in $U_G(\lambda)$ have Lie algebras that directly span $\mathrm{Lie}(U_G(\lambda))$, so the smooth connected k -subgroup they generate must equal $U_G(\lambda)$ (as $U_G(\lambda)$ is connected). But $B(\lambda) = T \ltimes U_G(\lambda)$, so $B(\lambda)$ is generated by T and the root groups U_a for those roots a whose weight space is contained in $\mathrm{Lie}(B(\lambda))$. ■

Within the theory of root systems, there is a concept of *positive system of roots*: these turn out to be exactly the sets of roots cut out by the condition $\langle a, \lambda \rangle > 0$ for linear forms λ on $X_{\mathbf{Q}}$ that are non-vanishing on all roots. It is a general fact that the Weyl group of the root system simply transitively permutes the set of such positive systems. But in the case of a split pair (G, T) we just saw in (the proof of) Proposition 4.16 that such positive systems Φ^+ in $\Phi(G, T)$ are exactly the sets of roots that occur in the Lie algebra of a Borel subgroup containing T . Indeed, we proved that the Lie algebra of *some* Borel subgroup has this form, and hence all do by the transitive $W(G, T)$ -action on the set of Borel subgroups and the evident fact that the $W(G, T)$ -action on $X(T)$ preserves $\Phi(G, T)$.

Now choose $w \in W(G, T)$. We will prove $w \in W(R(G, T))$. By the definitions, clearly $\Phi(w.B) = w.\Phi(B)$. Since $W(R(G, T))$ acts (simply) transitively on the set of all positive systems of roots in $\Phi(G, T)$ (by general facts in the theory of root systems), it follows that there exists w' in the subgroup $W(R(G, T))$ such that $w.\Phi(B) = w'.\Phi(B)$, so $\Phi(w^{-1}w'.B) = \Phi(B)$. By Proposition 4.16 this forces $w^{-1}w'.B = B$, and hence (by Proposition 1.8) $w = w'!$ Thus, the Weyl group of (G, T) is exactly the Weyl group of the associated root datum (or root system). And we have even proved a bonus: the set of Borel subgroups containing T is in natural bijective correspondence with the set of positive systems of roots in the root system. This is the first indication of how the subgroup structure of G in relation to T can be expressed in terms of the root datum and even be *understood* via general results in the combinatorial theory of root systems.