1. A map of abelian varieties \( f : A \to B \) over \( k \) is an \textit{isogeny} if it is surjective with finite kernel on \( \mathbb{F} \)-points, or equivalently (by generic flatness and translations) if \( f \) is finite and flat.

   (i) Using the theorem of Deligne from Exercise 5 in HW4 and the “quotient” property for fpqc homomorphisms between group schemes (as discussed in class), prove that if \( \dim A = \dim B \) then \( f \) is an isogeny if and only if there exists \( g : B \to A \) such that \( g \circ f = [n]_A \), in which case \( f \circ g = [n]_B \).

   (ii) Let \( \ell \) be a prime with \( \ell \neq \text{char}(k) \). Prove that \( f \) is an isogeny if and only if the induced map \( T_\ell(f) : T_\ell(A) \to T_\ell(B) \) on \( \ell \)-adic Tate modules is injective with finite cokernel, and equivalently if and only if \( V_\ell(f) : V_\ell(A) \to V_\ell(B) \) is an isomorphism. In such cases, prove that \( \deg f \) is not divisible by \( \ell \) if and only if \( T_\ell(f) \) is an isomorphism. (There are analogues for \( \ell = \text{char}(k) > 0 \), using Dieudonné modules.)

   (iii) The \textit{isogeny category} of abelian varieties over \( k \) has objects the abelian varieties over \( k \) and morphisms \( \text{Hom}^0(A,B) := \mathbb{Q} \otimes \mathbb{Z} \text{Hom}_k(A,B) \). Explain why this forms a category, prove that the “forgetful” functor from the category of abelian varieties over \( k \) to the isogeny category is faithful but not fully faithful, and that a map of abelian varieties is an isomorphism in the isogeny category if and only if it is an isogeny.

2. Let \( S \) be a scheme, \( X \) an \( S \)-scheme, and \( G \) an \( S \)-group scheme. Assume there is given a left action map \( G \times_S X \to X \). This action is called free if \( G(T) \) acts freely on \( X(T) \) for all \( S \)-schemes \( T \).

   (i) Prove that freeness is equivalent to the map \( G \times_S X \to X \times_S X \) defined by \( (\gamma, x) \mapsto (\gamma x, x) \) being a monomorphism, and deduce that freeness is insensitive to fpqc base change. (Hint: in a category with fiber products, a map is a monomorphism if and only if its relative diagonal is an isomorphism.)

   (ii) Let \( X \) be a scheme locally of finite type over an algebraically closed \( k \), equipped with an action by a \( k \)-group \( G \) locally of finite type. For each \( x \in X(k) \), prove that the functor assigning to any \( k \)-scheme \( S \) the subgroup of \( g \in G(S) \) fixing \( x \) is represented by a closed \( k \)-subgroup \( G_x \), the \textit{isotropy group scheme} at \( x \). Explain why \( G_x \) naturally acts on the tangent space \( T_x(X) \) (viewed as an affine space over \( k \)), so the action map \( G_x \to \text{GL}(T_x(X)) \) defines a map of Lie algebras \( \text{Lie}(G_x) \to \mathfrak{g}(T_x(X)) \) (i.e., an action in the sense of Lie algebra representations of \( \text{Lie}(G_x) \) on \( T_x(X) \)).

   Prove that the action is free if and only if \( G(k) \) acts freely on \( X(k) \) and \( \text{Lie}(G_x) \) acts freely on \( T_x(X) \) for all \( x \in X(k) \) (i.e., nonzero elements of \( \text{Lie}(G_x) \) act without nonzero fixed points on \( T_x(X) \)).

   (iii) Assume \( G \to S \) is fpqc and \( G \) acts freely on \( X \). A \textit{quotient} of \( X \) by the \( G \)-action is the \( G \)-invariant fpqc map \( \pi : X \to \overline{X} \) such that \( G \times_S X \to X \times_S \overline{X} \) defined by \( (g,x) \mapsto (gx,x) \) is an isomorphism. Prove that such a quotient, if it exists, is unique up to unique isomorphism, is initial among \( G \)-invariant maps from \( X \) to \( S \)-schemes, and retains the quotient property after base change to any \( S \)-scheme.

3. Let \( A \) be an abelian variety over \( k \), and \( G \) a \textit{finite} \( k \)-subgroup scheme of \( A \). This exercise proves the existence and uniqueness of a quotient abelian variety \( A/G \), and considers an important example.

   (i) Prove that up to unique isomorphism there is at most one pair \( \overline{A}, \pi \) consisting of an abelian variety \( \overline{A} \) and a surjective \( k \)-homomorphism \( \pi : A \to \overline{A} \) with \( G = \ker \pi \). Prove that if it exists then it is necessarily a quotient in the strong sense of Exercise 2(iii). Conversely, prove that if there is a quotient \( A/G \) in the strong sense of Exercise 2(iii) then it is necessarily an abelian variety. (Hint: a noetherian ring is regular if it admits a faithfully flat regular extension, by Theorem 23.7 of Matsumura CRT, and a \( k \)-algebra is finite type if it admits a faithfully flat extension of finite type over \( k \), by Prop. 9.1 in Exposé V of SGA3.)

   (ii) Choose \( n \in \mathbb{Z} - \{0\} \) killing \( G \) (e.g., the order of \( G \)), and consider the quotient mapping \( [n]_A : A \to A \) that identifies \( A \) with \( A/A[n] \) (in particular, \( A/A[n] \) exists and is an abelian variety). Explain how this identifies the problem of existence of \( A/G \) in the sense of (i) with the quotient problem from Exercise 2(iii) for the action of the \( A \)-group \( G \times A \) on \( A \) viewed as an \( A \)-scheme via \( [n]_A : A \to A \). The existence of quotients of free actions by finite flat group schemes on schemes affine (even finite!) over a noetherian base is solved in general by Theorem 4.1 in Exposé V of SGA3 (you can read §1–§4 there without the earlier exposés.)

   (iii) Let \( \mathcal{L} \) be an ample line bundle on \( A \), so \( K(\mathcal{L}) \) is a finite subgroup scheme of \( A \). Deduce that the dual abelian variety \( \hat{A} \) is naturally identified with the quotient \( A/K(\mathcal{L}) \). (In Mumford’s book, he develops from scratch a good theory of quotients of abelian varieties modulo finite subgroup schemes and then proves directly for ample \( \mathcal{L} \) that the quotient \( A/K(\mathcal{L}) \) satisfies the required properties to be a dual abelian variety. In this way he constructs the theory of the dual abelian variety without using the theory of Picard schemes.)