1. Let $X$ be a scheme over a field $k$, and assume that $X(k)$ is dense in $X$ (e.g., $k = k_s$ with $X$ geometrically reduced and locally of finite type). Prove that $X(k)$ is “relatively schematically dense” in $X$ in the following sense: for any $k$-scheme $S$, if a closed subscheme $Z$ of $X_S$ contains all sections in $X_S(S) = X(S)$ arising from $X(k)$ then $Z = X_S$.

2. Let $f : X \to S$ be a proper flat surjective map of schemes, with $S$ locally noetherian, and assume that the geometric fibers of $f$ are reduced and connected. Assume projectivity and/or noetherian hypotheses if you wish (but not needed).
   (i) Using the theory of cohomology and base change, prove that $\mathcal{O}_S = f_* \mathcal{O}_X$ via the natural map.
   (ii) Let $\mathcal{L}$ be a line bundle on $X$. Prove that $\mathcal{L} \simeq f^*(\mathcal{N})$ for a line bundle $\mathcal{N}$ on $S$ if and only if $f_*(\mathcal{L})$ is invertible and the natural map $f^* f_*(\mathcal{L}) \to \mathcal{L}$ is an isomorphism (in which case $\mathcal{N} \simeq f_* \mathcal{L}$). Deduce that in such cases, the formation of $f_*(\mathcal{L})$ commutes with any base change on $S$.

3. Let $Y$ be a normal locally noetherian separated scheme, and $U$ a dense affine open in $Y$. Prove that $Y - U$ has pure codimension 1 in the sense that its generic points have codimension 1 in $Y$ (i.e., local ring of $Y$ at such points is 1-dimensional).

4. Let $X \to S$ be a map of schemes and $Z \subseteq Z'$ a containment of $S$-flat closed subschemes whose associated ideal sheaves are locally finitely generated. If $Z_s = Z'_s$ inside of $X_s$ for all $s \in S$ then prove that $Z = Z'$ inside of $X$. (Hint: rename $Z'$ as $X$, and use Nakayama’s Lemma.)

5. Let $X$ be a smooth proper and geometrically connected curve of genus $g$ over a field $k$ such that $X(k) \neq \emptyset$, and let $P = \text{Pic}_{X/k}$ be its Picard scheme. We have seen in earlier homework that the $k$-group scheme $P$ is smooth of dimension $\dim H^1(X, \mathcal{O}_X) = g$ over $k$ and that $P^0$ is proper, so $P^0$ is an abelian variety of dimension $g$. In this exercise we identify $P^0(k)$ as a subgroup of $P(k) = \text{Pic}(X)$.
   (i) For any $k$-scheme $S$ and section $x \in X(S) = X_S(S)$, prove that the quasi-coherent ideal sheaf of the closed subscheme $x : S \hookrightarrow X_S$ is an invertible sheaf whose local generators are nowhere zero divisors on $\mathcal{O}_{X_S}$.
   (ii) For a coherent sheaf $\mathcal{F}$ on a proper $k$-scheme $Y$, recall that the Euler characteristic $\chi(\mathcal{F})$ is defined to be $\sum (-1)^i h^i(Y, \mathcal{F})$. For an invertible sheaf $\mathcal{L}$ on $X$, prove that $\chi(\mathcal{L}^n) = d_{\mathcal{L}} \cdot n + (1 - g)$ for an integer $d_{\mathcal{L}}$; we call this integer the degree of $\mathcal{L}$. Likewise, for a Weil divisor $D = \sum n_i x_i$ on our curve $X$, define $\text{deg}(D) = \sum n_i [k(x_i) : k]$. Prove that both notions of degree are invariant under extension of the ground field, and that they coincide when $Y = X$ and $\mathcal{L} \simeq \mathcal{O}_X(D)$.
   (iii) Choose $e \in X(k)$, and define $X^g \to P$ by defining $X(S)^g \to P(S) = \text{Pic}(X_S)/\text{Pic}(S)$ for any $k$-scheme $S$ to be
   $$\{(x_1, \ldots, x_g) \mapsto \mathcal{O}_{X_S}(x_1) \otimes \cdots \otimes \mathcal{O}_{X_S}(x_g) \otimes \mathcal{O}_{X^g}(e)^{\otimes (-g)}\}.$$ This map carries $(e, \ldots, e)$ to $0 \in P^0(k)$, so by connectedness of $X^g$ this map factors through a map $X^g \to P^0$ between proper $k$-schemes. Using the Riemann-Roch theorem for $X_{\overline{k}}$, prove that this latter map on $\overline{k}$-points hits exactly the line bundles on $X_{\overline{k}}$ of degree 0; don’t ignore the case $g = 0$.
   (iv) It is a general fact (proved in Ch. II, §5) that the Euler characteristic is locally constant for a flat coherent sheaf relative to a proper morphism of locally noetherian schemes. Deduce that there is a well-defined map of $k$-group schemes from $P$ to the constant group $\mathbf{Z}$ over $\text{Spec} \ k$ assigning to any point of $P(S)$ the locally constant function given by the fiberwise degree of the line bundle. Using that $\mathbf{Z}$ as a $k$-scheme contains no nontrivial $k$-proper subgroups, prove that for any field $K$, $P^0(K)$ is the subgroup of degree-0 line bundles in $\text{Pic}(X_K)$. (This depends crucially on the hypothesis that $X(k) \neq \emptyset$; Grothendieck gave a way to define $P = \text{Pic}_{X/k}$ without such a hypothesis on $X$, and then $P^0(k)$ can fail to have this concrete description when $\text{Br}(k) \neq 1$.)