Let $X$ be a projective scheme over a global field $K$, and let $\mathcal{L}$ be a line bundle on $X$. The base locus $B \subseteq X$ of $\mathcal{L}$ is the support of the coherent sheaf

$$\text{coker}(\Theta_X \otimes_K \mathcal{H}^0(X, \mathcal{L}) \to \mathcal{L}) = \{x \in X \mid s(x) = 0 \text{ in } \mathcal{L}(x) \text{ for all } s \in \mathcal{H}^0(X, \mathcal{L})\}.$$ 

In other words, for the natural map $f : X - B \to \mathbf{P}(\mathcal{H}^0(X, \mathcal{L}))$ (given by $x \mapsto [s_0(x), \ldots, s_n(x)]$ for a $K$-basis $\{s_0, \ldots, s_n\}$ of $\mathcal{H}^0(X, \mathcal{L})$) and the canonical isomorphism $\theta : f^*(\Theta(1)) \simeq \mathcal{L}$, there is no strictly large open set in $X$ across which $\theta$ extends. In this handout, we aim to prove that $h_{K,\mathcal{L}}$ is bounded below on $(X - B)(\overline{K})$. Since $K$ will be fixed throughout the discussion, we write $h_{\mathcal{L}}$ instead of $h_{K,\mathcal{L}}$. (It is natural to try to relate $h_{\mathcal{L}}\big|_{(X-B)(\overline{K})}$ to the composition of $X - B \to \mathbf{P}(\mathcal{H}^0(X, \mathcal{L}))$ and a standard height on the latter projective space, but we do not address that here.)

Pick a pair of very ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X$ such that $\mathcal{L} \simeq \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, so $h_{\mathcal{L}} = h_{\mathcal{L}_1} - h_{\mathcal{L}_2}$ (as functions $X(\overline{K}) \to \mathbf{R}$ modulo bounded functions). Thus, we need to show for some specific representatives $H_{\mathcal{L}_i}$ of the equivalence class $h_{\mathcal{L}_i}$ that $H_{\mathcal{L}_1} \geq H_{\mathcal{L}_2} + c$ on $(X - B)(\overline{K})$, for some $c \in \mathbf{R}$. If $\{U_1, \ldots, U_n\}$ is an open cover of $X - B$ then it suffices to do this for each $U_i(\overline{K})$ separately, allowing the constant $c$ and the choice of representatives $H_{\mathcal{L}_i}$ to depend on $i$. Letting $s$ vary through a basis of $\mathcal{H}^0(X, \mathcal{L})$, the corresponding open sets $X_s = \{x \in X \mid s(x) \neq 0\}$ cover $X - B$. Hence, it suffices to work on $X_s$ for a fixed choice of nonzero $s \in \mathcal{H}^0(X, \mathcal{L})$.

Multiplication by $s$ defines an injection of sheaves $\mathcal{L}_2 \to \mathcal{L}_1$ and hence a $K$-linear injection $\mathcal{H}^0(X, \mathcal{L}_2) \to \mathcal{H}^0(X, \mathcal{L}_1)$. Let $\{\sigma_0, \ldots, \sigma_n\}$ be a $K$-basis of $\mathcal{H}^0(X, \mathcal{L}_2)$, so $\{s\sigma_i\}$ is a $K$-linearly independent set in $\mathcal{H}^0(X, \mathcal{L}_2)$ and thus extends to a $K$-basis

$$\{s\sigma_0, \ldots, s\sigma_n, \tau_1, \ldots, \tau_m\}$$

of $\mathcal{H}^0(X, \mathcal{L}_2)$. We use these specific ordered bases to define $H_{\mathcal{L}_1}$ and $H_{\mathcal{L}_2}$ as the restrictions to $X(\overline{K})$ of the standard height functions via the canonical embeddings

$$X \hookrightarrow \mathbf{P}(\mathcal{H}^0(X, \mathcal{L}_1)) \simeq \mathbf{P}^n_K, \quad X \hookrightarrow \mathbf{P}(\mathcal{H}^0(X, \mathcal{L}_2)) \simeq \mathbf{P}^{n+m}_K.$$ 

In other words, for $x \in X(\overline{K})$ we have

$$H_{\mathcal{L}_2}(x) = \frac{1}{[K' : K]} \sum_{i'} \max_{i} \log \|\sigma_i(x)\|_{v'}$$

and

$$H_{\mathcal{L}_1}(x) = \frac{1}{[K' : K]} \sum_{i'} \max_{i} \log \|s\sigma_i(x)\|_{v'}, \quad \max_{j} \log \|\tau_j(x)\|_{v'}$$

where $K'/K$ is a finite subextension of $\overline{K}$ such that $x \in X(K')$ (and $v'$ ranges over the places of $K'$). Here, it is understood that the norm $\|\cdot\|_{v'}$ on the 1-dimensional $K'$-vector space $\mathcal{L}(x) \otimes_{K(x)} K'$ is defined using a single choice of $K'$-basis $e_i$; the specific choice doesn’t matter due to the product formula; this is the same calculation used to justify the homogeneity of the standard height on projective spaces. We likewise take the $K'$-basis of $\mathcal{L}(x) \otimes_{K(x)} K'$ to be $e_1 \otimes e_2^*.

Since we are restricting attention to points $x \in X_s(\overline{K})$, we have $s(x) \neq 0$, so we can scale by $\|s(x)\|_{v'}$ throughout to get

$$H_{\mathcal{L}_1}(x) = \frac{1}{[K' : K]} \sum_{i'} \max_{i} \log \|\sigma_i(x)\|_{v'}, \quad \max_{j} \log \|\tau_j(x)/s(x)\|_{v'} \geq H_{\mathcal{L}_2}(x).$$

Thus, $H_{\mathcal{L}_1}(x) - H_{\mathcal{L}_2}(x) \geq 0$. 