1. Overview

Let $G$ be a connected reductive group over a field $k$, $S$ a maximal $k$-split torus in $G$, and $P$ a minimal parabolic $k$-subgroup of $G$. Let $N = N_G(S)$ and $Z = Z_G(S)$, so $N/Z$ is a finite étale $k$-group (as for any torus centralizer in any smooth affine group). The ordinary group $N(k)/Z(k)$ is called the relative Weyl group and is usually denoted $kW$. (Later it will naturally be identified with the “combinatorial” Weyl group of the root system attached to $(G, S)$.)

In class we saw the proof of the Bruhat decomposition: the natural map

$$kW := N(k)/Z(k) \to P(k)\setminus G(k)/P(k)$$

is bijective. In other words, if we choose $Z(k)$-coset representatives $n_w \in N(k)$ for each $w \in kW$ then every $P(k)$-double coset in $G(k)$ has the form $P(k)n_wP(k)$ for a unique $w \in kW$.

In this handout, we establish a geometric refinement of the Bruhat decomposition: we show that if $n, n' \in N(k)$ are distinct modulo $Z(k)$ then $PnP \neq Pn'P$ as locally closed subschemes of $G$. (Equivalently, $P(k)nP(k) \neq P(k)n'P(k)$ inside $G(k)$.) We also address the question of how the finite group $N(k)/Z(k)$ is related to the finite étale $k$-group $N/Z$, ultimately showing that every connected component of $N$ has a $k$-point.

2. Geometric Bruhat

As explained above, for $n, n' \in N(k)$ distinct modulo $Z(k)$, we wish to show that $(PnP)(\bar{k})$ and $(Pn'P)(\bar{k})$ are disjoint. Since $P = Z \rtimes U$ for the $k$-unipotent radical $U = \mathcal{B}_{w,k}(P)$ and $n$ and $n'$ normalize $Z$, we have

$$(PnP)(\bar{k}) = Z(\bar{k}) \cdot (UnU)(\bar{k})$$

and similarly for $n'$. Thus, if $(PnP)(\bar{k})$ and $(Pn'P)(\bar{k})$ have a common point then there exists $z \in Z(\bar{k})$ such that $z(UnU)(\bar{k})$ meets $(UnU)(\bar{k})$. But $Z$ normalizes $U$, so $z(Un)(\bar{k}) = (U(zn)(\bar{k}))$ and $zn \neq n'$ since $n$ and $n'$ are distinct modulo $Z(\bar{k})$ (as $N(k)/Z(k)$ injects into $(N/Z)(\bar{k}) \subseteq (N/Z)(\bar{k}) = N(\bar{k})/Z(\bar{k})$). Recall also from our study of minimal parabolic $k$-subgroups in class that we may choose a cocharacter $\lambda : \text{GL}_1 \to S$ such that $P = P_G(\lambda)$ and $Z_G(S) = Z_G(\lambda)$. Hence, it suffices to prove the following result applied over $\bar{k}$ upon renaming such a hypothetical $zn$ as $n$:

**Proposition 2.1.** Let $G$ be a smooth connected affine group over a field, $S$ a torus of $G$, and $\lambda : \text{GL}_1 \to S$ a cocharacter of $S$ such that $Z_G(\lambda) = Z_G(S)$. Let $N = N_G(S)$, $Z = Z_G(S) = Z_G(\lambda)$, $U = U_G(\lambda)$, and $P = P_G(\lambda) = Z \rtimes U$.

For $n, n' \in N(k)$, if $UnU$ meets $Un'U$ then $n = n'$.

**Proof.** Without loss of generality $k = \bar{k}$, and since the double cosets meet we must have $nu = u'n'$ for some $u, u' \in U(k)$. Hence, $n^{-1}n = n^{-1}(nu)u^{-1} = n'^{-1}u'n'u^{-1}$. We aim to eventually prove that $n'^{-1}n = 1$, so as a preliminary step we will investigate the structure of the solvable (even unipotent) smooth connected group $V := n'^{-1}Un'$.
Note that $Sn' = n'S$, so $V$ is normalized by $S$ (since $S \subset Z \subset P$ and $U$ is normal in $P$), so it is normalized by $\lambda$. Thus, the dynamic method applies to $V$ (or rather to the semidirect product $GL_1 \ltimes V$ defined via the $GL_1$-action on $V$ through $\lambda$), so we obtain an open subscheme

$$U_V(-\lambda) \times Z_V(\lambda) \times U_V(\lambda) \hookrightarrow V$$

via multiplication. But in the connected solvable case this open immersion is always an equality! (Indeed, in the commutative case it follows from the fact that a smooth connected group has no proper open subgroups, and in general one can bootstrap from the commutative case to the solvable case via the derived series and appropriate dimension induction: see Proposition 2.1.12 and its proof in “Pseudo-reductive groups”.) Hence, this says that we have an equality of schemes via multiplication

$$(V \cap U') \times (V \cap Z) \times (V \cap U) = V,$$

where $U' := U_G(-\lambda)$. By definition of $V$ we have $V \cap Z = n'^{-1}(U \cap Z)n' = 1$, so

$$(V \cap U') \times (V \cap U) = V$$

via multiplication. Thus, the element $n'^{-1}u'n' \in V(k)$ can be written as $u'_1u_1$ for $u'_1 \in U'(k)$ and $u_1 \in U(k)$.

Recall that $n'^{-1}n = n'^{-1}u'n'u^{-1}$, so this is equal to $u'_1(u_1u^{-1}) \in U'(k) \times U(k)$. Thus, $n'^{-1}n$ lies in $N(k) \cap (U'(k) \times U(k))$. We claim that the intersection of $N$ with the open subscheme $U' \times Z \times U$ of $G$ is equal to $Z$, so this would force $u'_1, u_1u^{-1} = 1$ and hence $n' = n$ as desired. It now remains to prove rather generally that for any smooth closed subgroup $H$ of $Z$ (such as $S$), the inclusion

$$(Z_G(H) \cap U') \times N_Z(H) \times (Z_G(H) \cap U) \subseteq N_G(H) \cap (U' \times Z \times U)$$

inside $G$ is an equality. (Indeed, $N_Z(H) = Z_G(H) \cap Z$ and if $H = S$ then by hypothesis $Z_G(S) = Z_G(\lambda) = Z$ and we know that $Z \cap U', Z \cap U = 1$, so we would be done.) We will prove the equality by computing with points valued in arbitrary $k$-algebras $R$.

Choose a $k$-algebra $R$ and points $u' \in U'(R)$, $z \in Z(R)$, and $u \in U(R)$ such that $u'zu$ lies in $N_G(H)(R)$, which is to say that it normalizes $H_R$ inside $G_R$. The aim is to prove that $u, u'$ centralize $H_R$ and $z$ normalizes $H_R$. Such properties are sufficient to check on $R'$-valued points for every $R$-algebra $R'$, so upon choosing an $R'$ and renaming it as $R$ (as we may do), the task is to prove that $u, u'$ centralize $H(R)$ and $z$ normalizes $H(R)$ inside $G(R)$. That is, if $h \in H(R)$ we want to show that it commutes with $u$ and $u'$ and also that $zhz^{-1} \in H(R)$.

Consider the automorphism $f$ of $H_R$ induced by conjugation by $u'zu \in N_G(H)(R)$. For all $h \in H(R)$,

$$u' \cdot zh \cdot h^{-1}uh = u'zhuh = f(h)u'zu = (f(h)u'f(h)^{-1}) \cdot f(h)z \cdot u.$$

But $H \subset Z$ and so it normalizes $U$ and $U'$. Hence, the outer terms in this equality visibly correspond to decompositions in the subset $U'(R) \times Z(R) \times U(R) \subset G(R)$ (inclusion via multiplication), so corresponding terms coincide. This says exactly that

$$u' = f(h)u'f(h)^{-1}, \quad zh = f(h)z, \quad h^{-1}uh = u.$$
As we vary $h$ through $H(R)$, $f(h)$ likewise sweeps out $H(R)$ (as $f$ is an automorphism of $H_R$), so the first and third equalities give that $u$ and $u'$ are centralized by $H(R)$, whereas the second says $zhz^{-1} = f(h)$, so $z$ normalizes $H(R)$ inside $G(R)$.

3. Split Weyl group

Now we study the relationship between $N(k)/Z(k)$ and $N/Z$. As a first step, we show:

**Lemma 3.1.** The finite étale $k$-group $N/Z$ is constant. Equivalently, the natural $\text{Gal}(k_s/k)$-action on $(N/Z)(k_s) = N(k_s)/Z(k_s)$ is trivial.

Note that the equality $(N/Z)(k_s) = N(k_s)/Z(k_s)$ rests on the $k$-smoothness of $Z$.

**Proof.** Choose $n \in N(k_s)$ and $\gamma \in \text{Gal}(k_s/k)$. We want to show that $\gamma(n)^{-1}n \in Z(k_s)$, as that says $\gamma(n)$ and $n$ have the same image in $N(k_s)/Z(k_s) = (N/Z)(k_s)$, so the triviality of the Galois action would be proved.

Consider the $k_s$-automorphism of $S_{k_s}$ defined by conjugation against the element $n \in N(k_s)$: this is $x \mapsto nzn^{-1}$. For any two split $k$-tori $T$ and $T'$, all $k_s$-homomorphisms $T_{k_s} \to T'_{k_s}$ are defined over $k$. Thus, $n$-conjugation on $S_{k_s}$ is defined over $k$, which is to say that this $k_s$-automorphism is equivariant with respect to the application of any $\gamma$. That is, for $x \in S(k_s)$ we have $\gamma(nzn^{-1}) = n\gamma(x)n^{-1}$, but $\gamma(nzn^{-1}) = \gamma(n)\gamma(x)\gamma(n)^{-1}$, so $\gamma(n)^{-1}n$ centralizes all such $x$ and hence $\gamma(n)^{-1}n \in Z(k_s)$ as desired.

Since the cosets of $Z_{k_s}$ inside $N_{k_s}$ are the connected components of $N_{k_s}$, the triviality of the Galois action in the preceding lemma says exactly that each of these components is defined over $k$ inside $N$. In other words, the connected components of $N$ are geometrically connected over $k$. Rather more subtle is that each of these components actually contains a $k$-point. That property is equivalent to the assertion that the inclusion $N(k)/Z(k) \hookrightarrow (N/Z)(k)$ is an equality; and it is most remarkable since $H^1(k, Z)$ is utterly mysterious. Let us now prove it:

**Proposition 3.2.** The natural map $N(k)/Z(k) \to (N/Z)(k)$ is surjective.

**Proof.** Let $W$ denote the constant finite $k$-scheme $N/Z$. The idea for proving that the subgroup $N(k)/Z(k) \subseteq W(k)$ is full is to show that $W(k)$ acts freely on a set whose resulting $N(k)/Z(k)$-action is transitive. Motivated by the bijective correspondence between the set of Borel subgroups containing a given maximal torus and the set of Weyl chambers in the associated root system (or equivalently the set of positive systems of roots) in the split case, together with the simply transitive action of the combinatorial Weyl group on the set of chambers (and the equality of this Weyl group with the “Weyl group” defined by the reductive group and its chosen maximal torus), we are led to consider the set $\mathcal{P}$ of minimal parabolic $k$-subgroups $P$ of $G$ that contain the maximal $k$-split torus $S$.

There is an evident action of $N(k)$ on $\mathcal{P}$, and we claim that it is transitive. For any $P, P' \in \mathcal{P}$ we know there exists $g \in G(k)$ such that $P' = gPg^{-1}$, so $S$ and $gSg^{-1}$ are maximal $k$-split tori in $P'$. But we have shown that in any parabolic $k$-subgroup of a connected reductive $k$-group, all maximal $k$-split tori are $k$-rationally conjugate. Thus, there exists $p' \in P'(k)$ such that $p'gSg^{-1}p'^{-1} = S$. Hence, $p'g \in N(k)$ and this element conjugates $P$ to $P'$. This proves the transitivity of the $N(k)$-action on $\mathcal{P}$, and it factors through
Since we know that any parabolic $k$-subgroup $P$ of $G$ containing $S$ necessarily contains $Z_G(S) = Z$.

Now it remains to define a free action of $W(k)$ on $\mathcal{P}$ that restricts to the above action of $N(k)/Z(k)$ on $\mathcal{P}$. The preceding lemma implies that $W(k) = W(k_s)$, and we know that $W(k_s) = N(k_s)/Z(k_s)$. For any $w \in W(k)$, choose a representative $n \in N(k_s)$ and consider $nP_kn^{-1}$ for $P \in \mathcal{P}$. This is a parabolic $k_s$-subgroup of $G_{k_s}$ with the same dimension as $P$, so if it is defined over $k$ then its $k$-descent must be a minimal parabolic $k$-subgroup of $G$ (as we know that the minimal parabolic $k$-subgroups of $G$ all have the same dimension, due to their $G(k)$-conjugacy). For any $\gamma \in \text{Gal}(k_s/k)$ we have $\gamma(n) = nz$ for some $z \in Z(k_s) \subset P(k_s)$, so

$$\gamma(nP_kn^{-1}) = \gamma(n)P_k\gamma(n)^{-1} = nP_kn^{-1}.$$ 

Thus, $nP_kn^{-1}$ descends to a minimal parabolic $k$-subgroup of $G$, and we may denote it as $w.P$ since clearly $nP_kn^{-1}$ depends on $n$ only through its $Z(k_s)$-coset (which in turn depends only on $w$). It is clear that $P \mapsto w.P$ is an action of $W(k)$ on $\mathcal{P}$, and its restriction to an action of $N(k)/Z(k)$ is obviously the action considered above.  \[\blacksquare\]