Math 249C. Rational points of flag varieties

Let G be a connected reductive group over a field k, and P a parabolic k-subgroup. The quotient G/P is projective, and it is natural to wonder if the map

$$G(k) \to (G/P)(k)$$

is surjective. When k is finite then we can settle this affirmatively by using Lang's theorem, in the form that $\mathrm{H}^1(k, P) = 1$ (as P is smooth and connected). Indeed, for any $x \in (G/P)(k)$ the fiber over x in G is a right P-torsor over k and hence corresponds to a class in $\mathrm{H}^1(k, P) = 1$. This vanishing says that the right P-torsor has a k-point; that point is an element of $G(k)$ mapping onto x. Hence, we now may and do assume k is infinite. In particular, by the unirationality of G over k, $G(k)$ is Zariski-dense in G.

In general there is no reason to expect that $\mathrm{H}^1(k, P)$ vanishes, but nonetheless, as in the case of relative Weyl groups ${}_kW = N_G(S)/Z_G(S)$ (for which the lack of a cohomological explanation was no obstacle to showing that $N_G(S)(k) \to ({}_kW)(k)$ is surjective), we shall show that $G(k) \to (G/P)(k)$ is surjective.

It suffices to find a dense open $U \subset G$ that maps isomorphically onto its open image in G/P. Indeed, the translates gU for $g \in G(k)$ actually *cover* G (since $G(k)$ must also be dense in $G_{\overline{k}}$, so for any $\xi \in G(\overline{k})$ the non-empty open set $\xi U_{\overline{k}}^{-1}$ contains some $g \in G(k)$, forcing $\xi \in (gU)_{\overline{k}}$). Consequently, we would get an open covering of G/P (namely, by the images of the gU's for $g \in G(k)$) over which there are sections, ensuring that all points in $(G/P)(k)$ arise from $G(k)$.

To construct such a U, recall that $P = P_G(\lambda)$ for some k-homomorphism $\lambda : \mathrm{GL}_1 \to G$. Thus, via multiplication we get a dense open subscheme

$$U_G(-\lambda) \times Z_G(\lambda) \times U_G(\lambda) = U_G(-\lambda) \times P_G(\lambda) \to G,$$

implying that $U_G(-\lambda) \to G/P_G(\lambda) = G/P$ is an open immersion.