Math 249C. Absolute irreducibility

1. Introduction

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of a connected reductive k-group, with $k = k_s$. Assume that ρ is irreducible (meaning that there is no nonzero proper subspace V' stable under the k-action). Since $G(k)$ is Zariski-dense in G, it is equivalent to say that V is irreducible as a representation of the abstract group $G(k)$.

The aim of this handout is to prove that in such situations ρ is *absolutely irreducible*, meaning that the action of $G_{\overline{k}}$ on $V_{\overline{k}} = \overline{k} \otimes_k V$ is irreducible (or equivalently, that $G(\overline{k})$ acts irreducibly on $V_{\overline{k}}$). This only has content when $k \neq \overline{k}$, which is to say that k is imperfect. We let $p = \mathrm{char}(k) > 0$, so $\overline{\mathbf{F}}_p \subseteq k$. In order to appreciate the subtlety of our goal, we begin with a counterexample when the reductivity hypothesis is dropped:

Example 1.1. Let k'/k be a nontrivial purely inseparable finite extension of fields, V' a finite-dimensional k'-vector space with dimension $d > 1$, and $G = \mathrm{R}_{k'/k}(\mathrm{SL}(V'))$. This is not reductive since the natural map $G_{k'} \to \mathrm{SL}(V')$, corresponding on points valued in a k'-algebra A' to the map

$$G(A') = \mathrm{SL}(V' \otimes_k A') = \mathrm{SL}(V' \otimes_{k'} (k' \otimes_k A')) \twoheadrightarrow \mathrm{SL}(V' \otimes_{k'} A'),$$

has a nontrivial smooth connected kernel that is unipotent (apply Proposition A.5.12 in "Pseudo-reductive groups" to $G_{\overline{k}} = \mathrm{R}_{(k' \otimes_k \overline{k})/\overline{k}}(\mathrm{SL}(V' \otimes_k \overline{k}))$). The k-group G is pseudo-reductive (see Proposition 1.1.10 in "Pseudo-reductive groups"), but this doesn't matter for our purposes. Moreover, G is generated by k-tori (see Propositions A.2.11 and 1.3.4 in "Pseudo-reductive groups").

Let V be the k-vector space underlying V'. There is a natural linear representation of G on V, since the inclusion of k-algebras $\mathrm{End}_{k'}(V') \hookrightarrow \mathrm{End}_k(V)$ induces an inclusion $j : G = \mathrm{R}_{k'/k}(\mathrm{SL}(V')) \hookrightarrow \mathrm{SL}(V)$ of algebraic subgroups of the unit groups of these algebras over k. (This inclusion uses that the k-linear determinant of a k'-linear endomorphism is $\mathrm{N}_{k'/k}$ applied to the k'-linear determinant.) Functorially, for any k-algebra R, the map j on R-points corresponds to the evident R-linear action on $V_R = V' \otimes_k R$ by the group $G(R) = \mathrm{SL}(V' \otimes_{k'} (k' \otimes_k R)) = \mathrm{SL}(V' \otimes_k R)$.

The representation of G on V remains irreducible over k_s. Indeed, extending scalars to k_s causes k' to be replaced with the k_s-algebra $k' \otimes_k k_s = k'_s$ that is a nontrivial purely inseparable finite extension of k_s, so it suffices to show that if $k = k_s$ then the G-action on V is irreducible. Since $G(k)$ is Zariski-dense in G (as $k = k_s$), it is equivalent to say that $G(k)$ acts irreducibly on V, or in other words that the abstract unit group $\mathrm{GL}(V')$ of the matrix algebra $\mathrm{End}_{k'}(V')$ over k' acts irreducibly on V' viewed as a k-vector space. Concretely, we're claiming that the natural action of $\mathrm{SL}_d(k')$ on ${k'}^d$ is irreducible even at the k-linear (rather than k'-linear) level.

Suppose $W \subseteq {k'}^d$ is a nonzero k-subspace that is stable under the action of $\mathrm{SL}_d(k')$. Since $d \geq 2$, $\mathrm{SL}_d(k')$ generates $\mathrm{Mat}_d(k')$ as a *k-algebra* (not just as a k'-algebra). Indeed, by considering the open cell we reduce to the case of $\mathrm{SL}_2(k')$ inside $\mathrm{Mat}_2(k')$, and for $x' \in k'$

the elements

$$\begin{pmatrix} x' & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(k')$$

have difference equal to $\mathrm{diag}(x', 0)$. We can likewise generate $\mathrm{diag}(0, x')$, so $\mathrm{SL}_2(k')$ generates $\mathrm{Mat}_2(k')$ as a k-algebra. It follows that W is stable under the natural scaling action of k'. In other words, such a W is necessarily a k'-subspace, so the irreducibility of ${k'}^d$ as a k'-linear representation of $\mathrm{SL}_d(k')$ forces $W = {k'}^d$ as desired.

Now we come to the main point: V is *not* absolutely irreducible as a representation of G. Indeed, we claim that the k'-vector space $V_{k'}$ is reducible as a $G_{k'}$-representation. The key observation is that $V_{k'} = V' \otimes_{k'} (k' \otimes_k k')$ on which the natural action of $G_{k'} = \mathrm{R}_{(k'\otimes_k k')/k'}(\mathrm{SL}(V' \otimes_k k'))$ respects the *nontrivial* filtration arising from the filtration of the local artin ring $k' \otimes_k k'$ by powers of its *nonzero* nilpotent maximal ideal. In down-to-earth terms, since the k'-points in $G_{k'}$ are Zariski-dense (as an imperfect field is infinite, and G is Zariski-open open in the affine space over k obtained from an affine space of matrices over k'), the assertion can be checked at the level of rational points: we're claiming that the natural action of $\mathrm{SL}_d(k' \otimes_k k')$ on $(k' \otimes_k k')^d$ is reducible as a k'-linear representation when $k' \otimes_k k'$ is viewed as a k'-vector space through the second tensor factor. Even as a $k' \otimes_k k'$-module, the action of $\mathrm{SL}_d(k' \otimes_k k')$-action respects the I-adic filtration for any ideal I of the commutative ring $k' \otimes_k k'$. Taking I to be the nonzero nilpotent maximal ideal of $k' \otimes_k k'$ then provides the desired reducibility.

We have to show that phenomena as in the preceding example cannot arise when working with connected reductive groups. The key fact we will use about connected reductive k-groups (with $k = k_s$) is that they descend to the algebraically closed subfield $\overline{\mathbf{F}}_p \subseteq k$ (though we will use more basic properties too, such as that Cartan subgroups are tori).

2. The connected reductive case

Now suppose that G is connected reductive, so it is generated by its maximal k-tori. The k-algebra $\mathrm{End}_G(V)$ is a (not necessarily central) division algebra of finite dimension over k. Since $k = k_s$, this is a finite extension field k'/k. Thus, V is naturally a k'-vector space on which $G(k)$ acts k'-linearly. Write V' to denote V equipped with this k'-linear structure. The extension k'/k is purely inseparable since $k = k_s$.

Since the representation of $G(k)$ on V is k'-linear, the k-homomorphism $\rho : G \to \mathrm{GL}(V)$ factors through the k-subgroup $\mathrm{R}_{k'/k}(\mathrm{GL}(V'))$, and the resulting k-homomorphism $\rho : G \to \mathrm{R}_{k'/k}(\mathrm{GL}(V'))$ corresponds to a k'-homomorphism $\rho : G_{k'} \to \mathrm{GL}(V')$. Any k'-linear endomorphism L' of V' that commutes with the $G_{k'}$-action can be viewed as a k-linear endomorphism of V that commutes with the $G(k)$-action, so it lies in $\mathrm{End}_G(V)$ (as $G(k)$ is Zariski-dense in G), which is to say $L' \in k'$ inside $\mathrm{End}_k(V)$. Thus, $\mathrm{End}_{G_{k'}}(V') = k'$, which is to say that the k'-subalgebra A' of $\mathrm{End}_{k'}(V')$ generated by $G(k')$ has centralizer k'. But V' is simple as an A'-module (since any nonzero proper A'-submodule would also be a nonzero proper k-subspace of V stable under the action of $G(k)$ and hence of G, contradicting the irreducibility of V as a representation of G), so by Wedderburn's double centralizer theorem (Corollary 3.5, Chapter XVII in Lang's *Algebra*, 3rd edition) it follows that $A' = \mathrm{End}_{k'}(V')$. That is,

$G(k')$ generates $\mathrm{End}_{k'}(V')$ as a k'-algebra, so V' is absolutely irreducible as a representation of $G_{k'}$ over k'. Thus, our problem is exactly to show that $k' = k$.

Choose a maximal k-torus T in G. Since $k = k_s$, T is k-split. The k'-torus $T_{k'}$ acts k'-linearly on V', having associated weights $\chi'_1, \ldots, \chi'_d$ (counted with multiplicity). Since T is k-split, these χ'_i arise from k-rational characters χ_i of T. Hence, the characteristic polynomial of the representation of $G_{k'}$ on V' has restriction to $T_{k'}$ given by the monic polynomial $\prod(X - \chi'_i) \in k'[T][X]$ with coefficients in the coordinate ring $k'[T]$ of $T_{k'}$. This arises from the monic polynomial $f = \prod(X - \chi_i) \in k[T][X]$.

Lemma 2.1. *Let G be a connected reductive group over an infinite field. The set of semisimple points in $G(k)$ is Zariski-dense.*

Proof. Let T be a maximal k-torus of H. Consider the map $\pi : (G/T) \times T \to G$ defined by $(\overline{g}, t) \mapsto gtg^{-1}$. The image consists entirely of semisimple elements, and we claim it is dominant, so the k-points of a dense open inside the image would constitute a Zariski-dense set of semisimple elements in $G(k)$. To see that π is dominant, since its source and target are varieties of the same dimension it follows from semi-continuity of fiber dimension that π is dominant if it has a finite fiber. It is harmless to increase the ground field to be algebraically closed and *not* algebraic over a finite field. Thus, there exists $t_0 \in T(k)$ that generates T as a k-group, so the fiber $\pi^{-1}(t_0)$ consists of points $(\overline{g}, t)$ such that $gtg^{-1} = t_0$, which forces $g^{-1}t_0 g \in T$, so $g^{-1}Tg = T$ (since t_0 generates T). But then $g \in N_G(T)$, and $N_G(T)/T$ is finite since $T = Z_G(T)$ (as G is connected reductive). ∎

Lemma 2.2. *Let H be a connected reductive k-group, and let $\rho : H \to \mathrm{GL}(W)$ be an absolutely irreducible representation. If T is a maximal k-torus of H then the isomorphism class of ρ is determined by the restriction of the characteristic polynomials to T.*

Proof. Let $\rho' : H \to \mathrm{GL}(W')$ is an absolutely irreducible representation such that its characteristic polynomials have the same restriction to T as ρ does. In particular, W and W' have the same dimension. We seek to show that W and W' are H-equivariantly isomorphic.

Suppose there is such an isomorphism over $\overline{k}$. By Burnside's theorem, $H(\overline{k})$ generates $\mathrm{End}_{\overline{k}}(W_{\overline{k}})$ due to the absolute irreducibility. Thus, the scheme $\mathrm{Aut}_H(W)$ of H-equivariant automorphisms of W is equal to $\mathbf{G}_m$, so the scheme $\mathrm{Isom}_H(W, W')$ of H-equivariant isomorphisms is a $\mathbf{G}_m$-torsor over k (as it has a $\overline{k}$-point and so is non-empty). Such a torsor has a rational point, by Hilbert 90.

Now we may and do assume k is algebraically closed. By the Brauer-Nesbitt theorem, the isomorphism class of an irreducible finite-dimensional k-linear representation of an abstract group is determined by its characteristic polynomials. In our situation, the coefficients the the characteristic polynomials are regular functions on H. Thus, they are determined by their restriction to a dense set of k-points inside H. By Lemma 2.1, the $H(k)$-conjugates of $T(k)$ exhaust the points of a dense subset of H, so we're done. ∎

We conclude that the isomorphism class of V' as a $G_{k'}$-representation is uniquely determined by the property that the characteristic polynomial for the action of $T_{k'}$ is given by $f_{k'}$. Suppose for a moment that we can construct an absolutely irreducible representation W of G over k for which T acts with characteristic polynomial f. The preceding lemma implies that $W_{k'}$ is k'-linearly isomorphic to V' as a $G_{k'}$-representation. But the original k-linear representation of G on the underlying k-vector space V of V' is intrinsically recovered via the composition

$$G \hookrightarrow \mathrm{R}_{k'/k}(G_{k'}) \to \mathrm{R}_{k'/k}(\mathrm{GL}(V')) \hookrightarrow \mathrm{GL}(V)$$

where the final inclusion is induced by the inclusion of k-algebras $\mathrm{End}_{k'}(V') \subseteq \mathrm{End}_k(V)$. Hence, this k-linear representation of G is isomorphic to the k-linear representation given by the composition

$$G \hookrightarrow \mathrm{R}_{k'/k}(G_{k'}) \to \mathrm{R}_{k'/k}(\mathrm{GL}(W_{k'})) \hookrightarrow \mathrm{GL}_k(W_{k'})$$

where the final target denotes the k-group of linear automorphisms of the $W_{k'}$ viewed as a k-vector space. But this is a direct sum of $[k' : k]$ copies of W as a G-representation, so the assumed irreducibility of V as a k-linear representation of G forces $[k' : k] = 1$, which is to say $k' = k$ as desired.

It remains to prove the following descent result.

Proposition 2.3. *Let K'/K be an extension of fields, and G a split connected reductive K-group. Every absolutely irreducible representation $\rho' : G_{K'} \to \mathrm{GL}(W')$ descends to an absolutely irreducible representation (W, ρ) of G over K, unique up to isomorphism.*

We only need the case when K is separably closed, for which the intervention of Brauer groups in the proof can be easily avoided.

Proof. Let T be a split maximal K-torus of G. The action of $T_{K'}$ on $G_{K'}$ has some weights $\chi'_1, \ldots, \chi'_d$ on W', so the characteristic polynomial on $T_{K'}$ is $f' = \prod(X - \chi'_i) \in K'[T][X]$. But $\mathrm{X}(T_{K'}) = \mathrm{X}(T)$ since T is split, so $\chi'_i = (\chi_i)_{K'}$ for a K-rational character χ_i of T. Hence, $f' = f_{K'}$ for $f = \prod(X - \chi_i) \in K[T][X]$. Any descent to K must have T acting with characteristic polynomial f, so by Lemma 2.2 we see that such a descent is unique up to isomorphism if it exists.

By the Existence and Isomorphism Theorems for split connected reductive groups, we can choose a descent (G_0, T_0) of (G, T) to a split connected reductive group (equipped with a split maximal torus) over the prime field $\mathbf{F} \subset K$. Each χ_i descends to an $\mathbf{F}$-rational character $\chi_{i,0}$ of T_0, so $f_0 := \prod(X - \chi_{i,0}) \in \mathbf{F}[T_0][X]$ descends f_0. We seek to construct an absolutely irreducible representation W_0 of G_0 over $\mathbf{F}$ for which T_0 acts with characteristic polynomial f_0, as then Lemma 2.2 implies that $(W_0)_K$ is the desired descent of W' to K as a representation of G. For this purpose it is harmless to increase K' to be algebraically closed, so it contains an algebraic closure F of the prime field $\mathbf{F}$.

By Burnside's theorem, the abstract group $G(K') = G_0(K')$ generates $\mathrm{End}_{K'}(W')$ as a K'-algebra. Thus, there are *finitely many* elements of $G_0(K')$ that consistute a K'-basis of $\mathrm{End}_{K'}(W')$. Hence, by expressing K' as a direct limit of its finitely generated F-subalgebras A, we can find a sufficiently large A and a representation of $(G_0)_A$ on a finite free A-module $\mathcal{W}$ so that (i) $\mathcal{W}_{K'} \simeq W'$ as $G_{K'}$-representations, (ii) the weight space decomposition of W' for the action of $T_{K'} = (T_0)_{K'}$ descends to a decomposition of $\mathcal{W}$ as a direct sum of free A-modules of rank 1 with respective weights $(\chi_{i,0})_A$ for $(T_0)_A$, (iii) $G_0(A)$ generates $\mathrm{End}_A(\mathcal{W})$ as an A-algebra.

Passing to the quotient by a maximal ideal of A yields a representation σ of $(G_0)_F$ over F on which $(T_0)_F$ has characteristic polynomial $(f_0)_F = \prod(X - (\chi_{i,0})_F) \in F[T_0][X]$ and $G_0(F)$ generates the F-linear endomorphism algebra of the representation space of σ. This latter condition ensures that the F-specialization σ is absolutely irreducible, and it provides a descent of W' to F as a representation of $(G_0)_F$. If K were separably closed, so $F \subseteq K$ inside K', we would now be done. The rest of the argument handles the possibility that K may not be separably closed.

Clearly σ descends to an absolutely irreducible representation of G_0 over some finite Galois extension $\mathbf{F}'$ of the prime field, with the $(T_0)_{\mathbf{F}'}$-action having characteristic polynomial $(f_0)_{\mathbf{F}'}$. By Lemma 2.2, this descent is isomorphic to all of its $\mathrm{Gal}(\mathbf{F}'/\mathbf{F})$-twists since these twists are absolutely irreducible with the same characteristic polynomial for their $(T_0)_{\mathbf{F}'}$-actions (namely, $(f_0)_{\mathbf{F}'}$). In view of the absolute irreducibility, the obstruction to identifying the Galois twists in a manner that constitutes a descent datum and so provides descent from $\mathbf{F}'$ down to $\mathbf{F}$ lies in $\mathrm{Br}(\mathbf{F}'/\mathbf{F}) = \ker(\mathrm{Br}(\mathbf{F}) \to \mathrm{Br}(\mathbf{F}'))$. In positive characteristic we are done, since finite fields have trivial Brauer group.

It remains to consider the case of characteristic 0. In this case our task is to descend an absolutely irreducible representation of G_0 over $\overline{\mathbf{Q}}$ to a representation over $\mathbf{Q}$. We may assume $K' = \overline{\mathbf{Q}}$, and write G', T' to denote $(G_0)_{\overline{\mathbf{Q}}}$, $(T_0)_{\overline{\mathbf{Q}}}$. We first reduce to the semisimple case. It is harmless to replace G_0 with an isogenous cover, so we may assume $G_0 = Z_0 \times \mathscr{D}(G_0)$ for a split torus Z_0. The action of $(Z_0)_{K'}$ must be through a single character, since the action of G' must preserve each weight space for $(Z_0)_{K'}$ and by hypothesis this action is irreducible. Since any character of $(Z_0)_{K'}$ descends to a character of Z_0 over the prime field, we can twist it away so that Z_0 acts trivially. Thus, we can replace G_0 with its derived group, so it is semisimple.

By replacing G_0 with its simply connected isogenous cover, we may assume G_0 is simply connected. In characteristic 0, any linear representation of the Lie algebra of a simply connected semisimple group arises from a linear representation of the group itself. (This follows from a graph argument, using that in characteristic 0 any perfect Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_n$ arises from a connected closed subgroup H of GL_n over the ground field, with H necessarily semisimple if $\mathfrak{h}$ is semisimple.) Hence, our task is reduced to showing that for any split semisimple Lie algebra $\mathfrak{g}$ over a field k of characteristic 0 (such as $k = \mathbf{Q}$), its irreducible representations over $\overline{k}$ can be defined over k. Let $\mathfrak{a}$ be a choice of split Cartan subalgebra of $\mathfrak{g}$ over k, so we get a root space decomposition of $\mathfrak{g}$ over k. The finite-dimensional representations have weights for $\mathfrak{a}$ that must be “integral” in a sense described by the root system. The standard proof of the existence of finite-dimensional irreducible represenations over $\overline{k}$ with a given integral weight as its highest weight (relative to a choice of positive system of roots, or equivalently a Borel subalgebra containing $\mathfrak{a}$ over k) is carried out over k using the universal enveloping algebra of $\mathfrak{g}$ over k and the root space decomposition over k (and various integral weights of $\mathfrak{a}$, all defined over k). ■