Math 249B. Fiber dimension

Let $h : Z \to Y$ be a flat map of finite type between irreducible noetherian schemes, and let $\eta$ be the generic point of $Y$. By flatness, the generic point of $Z$ lies over that of $Y$, and the generic fiber $Z_\eta$ of finite type over $\eta$ is irreducible since it is a “localization” of $Z$. Let $d = \dim Z_\eta \geq 0$, so $Z_\eta$ has pure dimension $d$ (being irreducible of finite type over a field). We aim to prove that all non-empty fibers $Z_y$ have pure dimension $d$.

As a first step, we reduce to the case when $Y$ is the spectrum of a discrete valuation ring. We pick $y \in Y$ distinct from $\eta$ such that $Z_y$ is non-empty. By the Krull-Akizuki Theorem, there is a discrete valuation ring $R \subset k(Y)$ local over $\mathcal{O}_{Y,y}$, so the special fiber of $Z_R$ is a scalar extension of $Z_y$ to the residue field of $R$ (which could be a huge extension of $k(y)$). Since the property of a non-empty scheme of finite type over a field having a given pure dimension is insensitive to extension of the ground field, it is equivalent to show that $Z_R$ has special fiber of pure dimension $d$.

By design of $R$ inside $k(Y)$ it is clear that $Z_R \to \text{Spec}(R)$ has the same generic fiber $Z_\eta$ as for $Z$ over $Y$, and by $R$-flatness of $Z_R$ it follows that all generic points of $Z_R$ lie in its generic fiber $Z_\eta$. Thus, there is only one such point since the localization $Z_\eta$ of $Z_R$ is irreducible, so $Z_R$ is irreducible!

In this way, the base change to $R$ preserves our hypotheses (at the harmless cost of an extension of the residue field over $k(y)$), so we can replace $h$ with $Z_R \to \text{Spec}(R)$. In other words, we may now assume $Y$ is the spectrum of a discrete valuation ring $R$ and that the special fiber $Z_0$ is non-empty. Our task is to show that $Z_0$ has pure dimension $d$.

Since $Z_0$ is of finite type over the residue field $\kappa$ of $R$, to show it has pure dimension $d$ is a Zariski-local question near each closed point $z \in Z_0$. If some irreducible component $Z'_0$ of $Z_0$ has dimension distinct from $d$ then we may choose a closed point $z \in Z'_0$ not on any other irreducible component of $Z_0$, and then pick an affine open neighborhood of $z$ in $Z$ that meets $Z_0$ inside $Z'_0$. To get a contradiction we may replace $Z$ with that open neighborhood to arrive at the case when $Z_0$ is irreducible (and $Z$ is affine). In such cases it is enough to show that necessarily $\dim Z_0 = d$.

We may now drop the focus on “pure” dimension $d$: it is enough to show that a flat affine $R$-scheme of finite type with irreducible generic fiber of dimension $d$ (so the generic fiber is even of pure dimension $d$) and non-empty special fiber must have special fiber of dimension $d$. The following argument due to Artin is taken from [EGA IV$_3$, 14.3.10]. Since $Z$ is affine, if $n = \dim Z_0$ then we can choose a finite surjection $Z_0 \to \mathbf{A}_R^n$ by the Noether Normalization Theorem. By affineness this lifts to a map $Z \to \mathbf{A}_R^n$ over $R$. But this map is quasi-finite between the special fibers by design, and rather generally for any map of finite type $W \to V$ between noetherian schemes the locus of points $w \in W$ isolated in their fiber over $V$ (the “quasi-finite locus” of the morphism) is always open. By $R$-flatness of $Z$ the non-empty special fiber $Z_0$ cannot be open, so the open quasi-finite locus $\Omega \subset Z$ containing $Z_0$ must meet $Z_\eta$! Since $\Omega_\eta \to \mathbf{A}_{\mathcal{O}_Y}^n$ is quasi-finite, the non-empty $\Omega_\eta$ has dimension $\leq n$. But $\Omega_\eta$ is a non-empty open subset of $Z_\eta$ that has pure dimension $d$, so $d \leq n$ with equality if and only if $\Omega \to \mathbf{A}_{\mathcal{O}_Y}^n$ is dominant.

Suppose the quasi-finite $\Omega_\eta \to \mathbf{A}_{\mathcal{O}_Y}^n$ is not dominant (or equivalently, that $d < n$). Thus, $\Omega_\eta$ factors through a proper closed subset, and so through some hypersurface $(h = 0)$ for $h \in K[t_1, \ldots, t_n]$ of positive degree (with $K = \text{Frac}(R)$). We may scale $h$ by a unique integral power of a uniformizer of $R$ so that $h \in R[t_1, \ldots, t_n]$ and the reduction $h_0$ over $\kappa$ is nonzero (but possibly of lower degree than $h$ over $K$!). Then the preimage of $(h = 0)$ under $\Omega \to \mathbf{A}_{\mathcal{O}_Y}^n$ is a closed subscheme of $\Omega$ that contains $\Omega_\eta$. But $\Omega$ is an open subscheme of the $R$-flat $Z$, so it is also $R$-flat and hence the only closed subscheme of $\Omega$ containing $\Omega_\eta$ is $\Omega$. Thus, $\Omega \to \mathbf{A}_{\mathcal{O}_Y}^n$ factors through $(h = 0)$, so on special fibers over $\text{Spec}(R)$ we conclude that the map $\Omega_0 = Z_0 \to \mathbf{A}_R^n$ that is quasi-finite factors through $(h_0 = 0)$. But $(h_0 = 0)$ has dimension $n - 1$ when the nonzero $h_0$ is non-constant and it is empty when $h_0$ is constant, so $\dim Z_0 \leq n - 1$. This contradicts that $n = \dim Z_0$ with $Z_0$ non-empty.