Math 249B. Center and adjoint kernel

1. Introduction

It is a classical fact that if G is a connected Lie group then the kernel of the adjoint representation

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$$

coincides with the center Z_G. Indeed, by the definition in terms of the differential of conjugation it is obvious that $Z_G \subset \ker \mathrm{Ad}_G$, and for the reverse containment we need to check that if $g \in G$ has the property that the conjugation automorphism $c_g : x \mapsto gxg^{-1}$ of G induces the identity on $\mathfrak{g}$ then $c_g = \mathrm{id}_G$ (as that is exactly the centrality of g). But this in turn follows from the faithfulness of the Lie-algebra functor on connected Lie groups.

The same argument works for (necessarily separated) smooth connected groups G over a field k of characteristic 0: we just need to prove the faithfulness of the Lie-algebra functor for such G, and passage to the (generally non-commutative) formal group $\widehat{\mathscr{O}}_{G,e}$ is certainly faithful (as G is integral), so our task comes down to proving that on smooth formal groups in characteristic 0 the "tangent space" functor is faithful. But that in turn is a classical fact in the theory of formal groups; see Theorem 3 in §6 of Chapter V of Serre's book *Lie groups and Lie algebras* (wherein §6–§7 of Chapter IV and §5–§6 of Chapter V provide a self-contained development of formal groups culminating in that result).

The technique of faithfulness of the Lie-algebra functor breaks down in characteristic $p > 0$. Here is an instructive example in the unipotent case:

Example 1.1. Over $\mathbf{F}_p$, consider the alternating biadditive 2-cocycle $b : \mathbf{G}_{\mathrm{a}} \times \mathbf{G}_{\mathrm{a}} \to \mathbf{G}_{\mathrm{a}}$ defined by

$$b(x, y) = x^{p^2} y^p - y^{p^2} x^p.$$

Define U to be the non-commutative central extension of $\mathbf{G}_{\mathrm{a}}$ by itself via

$$(x, y) \cdot (x', y') = (x + x' + b(y, y'), y + y'),$$

with identity $(0, 0)$ and inversion $(x, y)^{-1} = (-x, -y)$. (The associativity of this composition law expresses the 2-cocycle condition on b.) This class of groups is inspired by the 2-dimensional non-commutative wound unipotent groups in Example B.2.9 of [CGP].

Assume $p \neq 2$. Since b is skew-symmetric and $p \neq 2$, the scheme-theoretic center Z_U is the functor of points (x, y) satisfying $y^p = 0$. But by direct computation of conjugation against points $(x\varepsilon, y\varepsilon)$ valued in the dual numbers $R[\varepsilon]$ for any k-algebra R with $x, y \in R$, we find that the adjoint representation of U is trivial!

Over $\mathbf{F}_2$ there is a variant of this construction that yields a non-commutative unipotent smooth connected affine group whose adjoint representation is trivial. The preceding construction is the "specialization at $a = 1$" for the construction in odd characteristic given in Example B.2.9 of [CGP] (2nd edition, as always). In that Example a modified (not as explicit) construction is made over the polynomial ring $\mathbf{F}_2[a]$ using Galois descent from $\mathbf{F}_4[a]$; the reduction modulo $a - 1$ furnishes the desired example over $\mathbf{F}_2$.

In view of the preceding example, to prove that $Z_G = \ker \mathrm{Ad}_G$ for a class of non-commutative smooth connected affine k-groups G in characteristic $p > 0$, we need to avoid the unipotent case and so are led to focus on the reductive case. Our aim is to prove that everything works well in that case:

Theorem 1.2. *If G is a connected reductive group over an arbitrary field k, then the inclusion $Z_G \subset \ker \mathrm{Ad}_G$ of closed k-subgroup schemes of G is an equality.*

Our proof of this result will be by characteristic-free methods, although the real substance is its validity in positive characteristic. As a special case of the theorem, $Z_G = 1$ if and only if the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ is faithful (i.e., has trivial schematic kernel, or equivalently is a closed immersion). This is the reason that such G with trivial center are called *adjoint type*.

Remark 1.3. With appropriate definitions for reductivity (with connected fibers) over a general base scheme, Theorem 1.2 is valid over any base; see Proposition 3.3.8 in the Luminy SGA3 notes (whose proof is a mild simplification of the proof of the essential content of Proposition 4.11 in Exp. XII of SGA3). The proof we give below is extracted from the argument given in the context of a general base scheme.

The fact that Ad_G is a closed immersion when $Z_G = 1$ also remains true over any base scheme (see Proposition 5.3.5 in the Luminy SGA3 notes), but this result even over $\mathbf{Z}$ or a discrete valuation ring lies *much* deeper than the case over fields (since a monic homomorphism between smooth affine groups need not be a closed immersion when the base is the affine line over a field of characteristic 0; see Example 3.1.2 in the Luminy SGA3 notes).

2. Proof of Theorem 1.2

We may and do assume k is algebraically closed. Rather generally, a closed normal subgroup scheme N in G is central if its identity component N^0 is central. Indeed, if N^0 is central then G/N^0 has center Z_G/N^0 (by the known good behavior with respect to passage to central quotients for the formation of the scheme-theoretic center of a connected reductive group), so to prove N is central in G it suffices to show that N/N^0 is central in G/N^0. Renaming the reductive quotient G/N^0 as G, we are reduced to the case that N is finite étale. The conjugation action by the *connected* G on the normal finite étale N is therefore trivial, which exactly expresses the centrality of N in G.

Returning to the situation of interest, setting $N = \ker \mathrm{Ad}_G$, it suffices to prove that $H := (\ker \mathrm{Ad}_G)^0$ is central in G. Note that H is normal in G since N is normal in G. (Normality can be checked using $G(k)$-conjugation since G is smooth; beware that H is generally not smooth.) Let $T \subset G$ be a maximal torus. By consideration of the T-equivariant and schematically dense open cell $\Omega \subset G$ associated to a choice of positive system of roots in $\Phi := \Phi(G, T)$, we see that

$$\ker(\mathrm{Ad}_G|_T) = \bigcap_{a \in \Phi} \ker a^\vee = Z_G.$$

Hence, it suffices to show that $H := (\ker \mathrm{Ad}_G)^0 \subset T$. Since T is its own scheme-theoretic centralizer in G, it is the same to prove that $H \subset Z_G(T)$, which is to say that T-conjugation on the connected normal subgroup scheme H is trivial.

In general, an action of a torus T on a connected k-group scheme H of finite type is trivial if the T-action on $\mathrm{Lie}(H)$ is trivial. This is proved via the complete reducibility of the finite-dimensional representation theory of T, applied to the T-action on the coordinate rings of the infinitesimal neighborhoods of the identity in H; see Corollary A.8.11 in [CGP] (proved for the action of a linearly reductive group scheme, such as a group scheme of multiplicative type, but whose proof simplifies in the relevant case of an action by a torus). Hence, it suffices to prove triviality of the T-action on

$$\mathrm{Lie}((\ker \mathrm{Ad}_G)^0) = \mathrm{Lie}(\ker \mathrm{Ad}_G) = \ker(\mathrm{Lie}(\mathrm{Ad}_G)) = \ker \mathrm{ad}_{\mathfrak{g}},$$

where $\mathrm{ad}_{\mathfrak{g}} : X \mapsto [X, \cdot]$ is the adjoint representation of $\mathfrak{g}$. The kernel of $\mathrm{ad}_{\mathfrak{g}}$ is certainly T-stable, so it is the span of its T-weight spaces. Hence, to show that only the trivial T-weight occurs it suffices to show that none of the T-root lines $\mathfrak{g}_a$ ($a \in \Phi(G,T)$) are killed by $\mathrm{ad}_{\mathfrak{g}}$; i.e., each such root line has nontrivial Lie bracket against something in $\mathfrak{g}$.

Passing to $\mathscr{D}(Z_G(T_a))$ and its maximal torus $T \cap \mathscr{D}(Z_G(T_a)) = a^\vee(\mathbf{G}_m)$ (whose root lines are $\mathfrak{g}_{\pm a}$) in place of G and T, we are reduced to the rank-1 semisimple case. That is, we can assume G is equal to either SL_2 or PGL_2, with T the diagonal torus and $\mathfrak{g}_a$ the upper-triangular root line. Let v^+ be a nonzero vector in that root line, v^- a nonzero vector in the opposite root line, and X a nonzero vector in $\mathrm{Lie}(T)$.

In the SL_2-case we have $[v^+, v^-] \neq 0$ by direct verification. In the PGL_2-case that calculation doesn't work out in characteristic 2, but we can instead use an alternative calculation that does work uniformly in all characteristics: since each root generates the character lattice of T (as PGL_2 has trivial center, or by inspection: the roots carry $\mathrm{diag}(t, 1)$ to $t^{\pm 1}$), we have $[X, v^+] = a'(X)v^+ \neq 0$ where

$$a' = \mathrm{Lie}(a) : \mathfrak{t} \simeq \mathrm{Lie}(\mathbf{G}_m) = k\partial_t|_{t=1} = k.$$