Math 249B. Nilpotence of connected solvable groups

1. Motivation and examples

In abstract group theory, the *descending central series* $\{C^i(G)\}$ of a group G is defined recursively by $C^0(G) = G$ and $C^{i+1}(G) = (G, C^i(G))$ for $i \geq 0$ (so $C^1(G)$ is the derived group, but for $i > 1$ typically $C^i(G)$ is much larger than the ith term of the derived series). This is a decreasing chain of normal (even characteristic) subgroups. One says that G is *nilpotent* if $C^i(G) = 1$ for $i \gg 0$. As with solvability (using the derived series), nilpotent is inherited by subgroups, quotients, and extensions. Note that if G is nilpotent and nontrivial, then the last nontrivial term $C^{i_0}(G)$ in the descending central series satisfies $(G, C^{i_0}(G)) = 1$, which is to say that $C^{i_0}(G)$ is a nontrivial *central* subgroup of G. In particular, in general $C^i(G)/C^{i+1}(G)$ is a central subgroup of $G/C^{i+1}(G)$.

It is a classical fact that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups. The reason for the "nilpotent" terminology is undoubtedly due to the following example:

Example 1.1. Let U be a unipotent linear algebraic group over a field k. The group $U(k)$ is nilpotent. Indeed, since we may choose a k-subgroup inclusion of U into the standard strictly upper triangular subgroup $\mathbf{U}_n$ in GL_n for some n, we may assume $U = \mathbf{U}_n$. Writing $U = 1 + N_n$ where N_n is the affine space of strictly upper triangular nilpotent $n \times n$ matrices, it is easy to check that the subgroups $U_i(k) := 1 + N_n(k)^i$ satisfy $(U_i(k), U_j(k)) \subset U_{i+j}(k)$ for all $i, j \geq 1$. Hence, $(U(k), U_j(k)) \subset U_{1+j}(k)$ for all $j \geq 1$, so by induction we see that $C^i(U(k)) \subset U_i(k)$ for all $i \geq 1$. But $N_n(k)^n = 0$, so $U_n(k) = 1$. Hence, $C^n(U(k)) = 1$.

We know from the previous course that if H, H' are smooth closed k-subgroups of a smooth finite type k-group G, there exists a unique smooth closed k-subgroup $(H, H') \subset G$ such that for every algebraically closed extension K/k the subgroup $(H, H')(K) \subset G(K)$ coincides with $(H(K), H'(K))$. In particular, much as with the derived series $\{\mathscr{D}^i(G)\}$ we can define the *descending central series* $\{\mathscr{C}^i(G)\}$ recursively via $\mathscr{C}^0(G) = G$ and $\mathscr{C}^{i+1}(G) = (G, \mathscr{C}^i(G))$ for $i \geq 0$. Computation with $\overline{k}$-points shows that each $\mathscr{C}^i(G)$ is normal in G,

The noetherian condition on G implies that the descending chain $\{\mathscr{C}^i(G)\}$ of closed k-subgroups stabilizes for some large i; we denote this final term as $\mathscr{C}^\infty(G)$. For any algebraically closed field K/k, the descending central series of $G(K)$ stabilizes at $\mathscr{C}^\infty(G)(K)$. Hence, $G(K)$ is nilpotent for all K/k if and only if it is nilpotent for one K/k, and this is equivalent to the condition $\mathscr{C}^\infty(G) = 1$. For this reason, we say that G is *nilpotent* if $\mathscr{C}^\infty(G) = 1$. Applying Example 1.1 on $\overline{k}$-points, we arrive at:

Proposition 1.2. *A unipotent linear algebraic group over a field is nilpotent.*

In general a connected solvable linear algebraic group need not be nilpotent. For example, consider $G = \mathbf{G}_m \ltimes \mathbf{G}_a$, the standard semi-direct product (using the $\mathbf{G}_m$-scaling action on $\mathbf{G}_a$); this is a Borel subgroup of PGL_2. It is not nilpotent since $\mathscr{C}^1(G) = \mathscr{D}(G)$ is the unipotent radical $U = \mathbf{G}_a$ and $(G, U) = U$ (so $\mathscr{C}^\infty(G) = U$). In §3, we will see that this example illustrates the "only" obstruction to nilpotence in the connected solvable case. However, we first need to digress and discuss an important property of unipotent group actions that was not addressed in the previous course.

2. Unipotent orbits

In general, over an algebraically closed field the orbits of a linear algebraic group G acting on a non-empty affine variety are locally closed, and the minimal-dimensional orbits are closed. But when G is unipotent, something remarkable happens:

Proposition 2.1. *Let U be a unipotent linear algebraic group over a field k, and X a quasi-affine k-scheme of finite type equipped with an action by U. The U-orbits $U.x$ for $x \in X(k)$ are closed.*

In the special case $U = \mathbf{G}_a$, this has a simple proof: we choose a projective closure $\overline{X}$ of X, so the orbit map $U \to X$ through $x_0 \in X(k)$ uniquely extends to a map $\mathbf{P}^1 \to \overline{X}$. This latter map has closed image, so restricting it over the open $X \subset \overline{X}$ gives that U has closed image except possibly when $\mathbf{P}^1$ lands inside X. But since X is quasi-affine, in such exceptional cases the image of $\mathbf{P}^1$ is a point, so the orbit map is constant. The general case does not seem to easily reduce to this case (e.g., by using a composition series of U), so unipotence has to be exploited in some other way (via its representation-theoretic consequences: Lie–Kolchin).

Proof. This result is proved as 4.10 in Borel's textbook, but his proof seems to have a small discrepancy that we clarify below. First, we recall two equivalent ways to define the notion of "quasi-affine scheme": by EGA II, 5.1.2, if S is an arbitrary quasi-compact scheme then it is equivalent that the natural map $S \to \mathrm{Spec}(\mathscr{O}(S))$ is an open immersion and that S is a quasi-compact open subscheme of an affine scheme. Such schemes are called *quasi-affine*. By EGA II, 5.1.9, if S is finite type over a ring R then it is quasi-affine if and only if it is an open subscheme of an affine R-scheme of finite type (the "classical" definition of quasi-affineness). We will use the "abstract" criterion in terms of open subschemes of $\mathrm{Spec}(\mathscr{O}(S))$ to circumvent the unclear step in 4.10 in Borel's book; note that even if S is finite type over a field k, $\mathscr{O}(S)$ is generally *not* finitely generated over k.

Define $Z \subset X$ to be the Zariski closure of the locally closed orbit $U.x$ of U through x, and let Z' denote the Zariski closure of Z under the open immersion $j : X \to X' := \mathrm{Spec}(\mathscr{O}(X))$ (so the affine scheme X' has coordinate ring $k[X']$ equal to $\mathscr{O}(X)$; typically X' is *not* of finite type over k). Clearly $F := Z - U.x$ is closed in the quasi-affine X, and we seek to prove that it is empty. Let $J \subset k[Z']$ be the radical ideal of elements that vanish on F, which is to say that it is the ideal of the Zariski closure F' of the reduced scheme F under the open immersion $j : X \to X'$. The k-point x in Z' does not belong to F' (as may be seen in the open subscheme X of X' which meets F' in F), so the k-point x is not in the zero scheme of J on the k-scheme X'. Hence, there exists $f \in J$ such that $f(x) \in k^{\times}$. In particular, $J \neq 0$.

The U-action on X canonically extends to an action on the affine scheme X' because for any k-algebra R the coordinate ring $R \otimes_k \mathscr{O}(X)$ of X'_R is naturally identified with the R-algebra $\mathscr{O}(X_R)$ of global functions on the base change X_R of the quasi-affine k-scheme X (since X is quasi-compact and separated over k, and R is k-flat). Here, we are using the functorial notion of "action" for a k-group scheme on an arbitrary affine k-scheme, as was introduced in the February 5 lecture of the previous course. But F and Z are U-stable closed subschemes of X, so their Zariski closures F' and Z' in X' are U-stable (since the formation of this Zariski closure commutes with base change to any k-algebra R, due to the k-flatness of R).

It follows from 12.1.1 of the February 5 lecture of the previous course (applied to the affine k-scheme Z' which may *not* be of finite type!) that the k-algebra $k[Z']$ is exhausted by a directed union of finite-dimensional k-subspaces V_α stable under the U-action. (The "error" in Borel's argument is that he works throughout with Z rather than Z', whereas it is Z' that is affine and Z is only quasi-affine, so he does not address the "algebraicity" of the action of U on $k[Z']$; our functorial arguments took care of this issue, since the February 5 lecture of the previous course was designed to apply to actions on arbitrary affine k-schemes without finiteness hypotheses.) Since $J \neq 0$, by choosing α sufficiently large we obtain $J_\alpha := J \bigcap V_\alpha \neq 0$. Thus, J_α is a nonzero finite-dimensional *algebraic* representation space for the unipotent linear algebraic group U. Any such representation can be upper triangularized over the ground field, by Lie–Kolchin, so $J_\alpha^U \neq 0$. That is, we obtain a nonzero U-invariant $h \in J \subset k[Z']$. The restriction of h to $U.x$ is equal to the constant $h(x) \in k$, and by the schematic density of $U.x$ in Z (and of Z in Z') it follows that $h = h(x) \in k$. But $h \neq 0$, so $h \in k^\times$. This says that the ideal J of the closed set F' in the affine scheme Z' meets $k^\times$, so F' is empty and hence F is empty. ∎

3. Characterization of nilpotence

Let G be a solvable connected linear algebraic group over a field k, and $T \subset G$ a maximal k-torus. Thus, $G_{\overline{k}} = T_{\overline{k}} \ltimes \mathscr{R}_u(G_{\overline{k}})$. Clearly this semi-direct product is a direct product if and only if T is central in G (as we may check centrality over $\overline{k}$), in which case G is certainly nilpotent. Indeed, in such cases $\mathscr{C}^i(G)_{\overline{k}} = \mathscr{C}^i(\mathscr{R}_u(G_{\overline{k}}))$ for all $i > 0$, and this vanishes for large i since $\mathscr{R}_u(G_{\overline{k}})$ is unipotent. Even in the general (connected) solvable case, we at least have $\mathscr{C}^1(G)_{\overline{k}} = \mathscr{D}(G)_{\overline{k}} \subset \mathscr{R}_u(G_{\overline{k}})$, so $\mathscr{C}^i(G)$ is unipotent for $i \geq 1$.

Theorem 3.1. *The k-group G is nilpotent if and only if T is central in G.*

Proof. We have proved the implication "$\Leftarrow$" above. The converse is rather more difficult to prove. We may and do assume $k = \overline{k}$, and we let $U = \mathscr{R}_u(G)$. The key point is to prove a general commutator result for connected solvable groups that has nothing to do with nilpotence. To formulate this, consider $t \in T(k)$ and the commutator morphism $c_t : U \to U$ defined by $u \mapsto (t, u) = (tut^{-1})u^{-1}$. This is the left t-translate of the orbit map for U through t under its conjugation action on G. Hence, by Proposition 2.1, $c_t(U)$ is a closed integral subscheme of U. It then makes sense to consider the scheme endomorphism $c_t : c_t(U) \to c_t(U)$. We claim that this is always an isomorphism. Once this is proved, it follows that $c_t(U) = c_t^n(U) \subset \mathscr{C}^n(G)$ for all $n \geq 1$, so $c_t(U) \subset \mathscr{C}^\infty(G)$. Hence, *if* G is nilpotent then $c_t(U) = 1$ for all $t \in T(k)$, which is exactly the assertion that T centralizes U and so is central in G.

Now we turn our attention to verifying that $c_t : c_t(U) \to c_t(U)$ is an isomorphism of schemes in the general connected solvable case. The proof given in 9.3(3) of Borel's textbook (which considers the weaker property of bijectivity on k-points) involves a long chain of reasoning for which it seems hard (at least for me) to get much intuition beyond checking that it works. But there is a very illuminating proof in characteristic 0 (to be explained below), and we will use a nontrivial result of Tits to show that the same technique can be adapted to work in positive characteristic.

First, we explain the toy example in any characteristic that motivates believing that $c_t : c_t(U) \to c_t(U)$ is an isomorphism (and explains why we work with $c_t(U)$ in the first place): suppose $G = T \ltimes V$ where the semi-direct product is taken with respect to a linear representation of T on a vector space V. In such cases V decomposes as a direct sum $\oplus V_\lambda$ of eigenspaces relative to the *semisimple* endomorphism defined by the action of t on V (recall that $k = \overline{k}$), and c_t acts on V as multiplication by $\lambda - 1$ on V_λ for each λ. Hence, $c_t(V) = \oplus_{\lambda \neq 1} V_\lambda$, so obviously $c_t : c_t(V) \to c_t(V)$ is an automorphism of the variety $c_t(V)$ (as it is multiplication by $\lambda - 1$ on each V_λ with $\lambda \neq 1$). The idea in general is to find a characteristic composition series of U that reduces the problem to the toy example. The subtlety is that c_t is generally not a homomorphism when U is not commutative, so an arbitrary T-equivariant composition series of U (e.g., the derived series) usually is ill-suited to carrying out a dimension induction.

To reduce the general case (in any characteristic) to the case with commutative U, we shall use the descending central series. We may assume $U \neq 1$ and that the general isomorphism property for c_t on $c_t(\mathscr{U})$ is known for all T-actions on unipotent connected linear algebraic k-groups $\mathscr{U}$ (commutative or not) when $\dim \mathscr{U} < \dim U$. Granting the commutative case, we may assume that U is not commutative. Since U is nilpotent (Proposition 1.2), the final nontrivial term U' in the descending central series of U is a *central* connected linear algebraic subgroup U' of U with $U' \subset \mathscr{C}^1(U) = \mathscr{D}(U) \neq U$. Consider the resulting short exact sequence

$$1 \to U' \to U \xrightarrow{q} U'' \to 1$$

in which U' and U'' each have strictly smaller dimension than U. This is T-equivariant due to the "characteristic" property of U' in U (more precisely, U' is stable under all k-automorphisms of U, and we can check T-equivariance by computing with k-points).

Since U' is commutative, the morphism $c'_t : U' \to U'$ is a homomorphism (namely, $c'_t(u') = t.u' - u'$ in additive notation, using the T-action on U' defined by conjugation). By dimension induction, c'_t restricts to an automorphism of the central subgroup $c'_t(U')$ in U. In view of the definition of the morphism of k-schemes $c_t : U \to U$, it factors through the central quotient map $q : U \to U''$; this corresponds to the map σ in the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "c_t"'] & U'' \arrow[dl, "\sigma"'] \arrow[d, "c''_t"] \\ c_t(U) \arrow[r, "q"'] & c''_t(U'') \end{tikzcd}$$

By hypothesis c''_t restricts to an *isomorphism* $c''_t(U'') \simeq c''_t(U'')$, so composing its inverse with $\sigma|_{c''_t(U'')}$ defines a map $s : c''_t(U'') \to c_t(U)$ that is a section to the natural map of integral algebraic k-schemes $q : c_t(U) \to c''_t(U'')$. This latter map is invariant under translation by the central subgroup $c_t(U) \bigcap U'$, so by multiplication we get a natural T-equivariant isomorphism of schemes

$$h : (c_t(U) \bigcap U') \times c''_t(U'') \to c_t(U).$$

In particular, this scheme isomorphism implies that the affine algebraic k-group scheme $c_t(U) \bigcap U'$ is smooth and connected (and visibly unipotent).

The isomorphism h intertwines c_t on $c_t(U)$ with the direct product map $c'_t \times c''_t$ on the source. By dimension induction we know that c''_t restricts to an automorphism of $c''_t(U'')$, so the automorphism property for c_t on $c_t(U)$ is equivalent to the property that c_t restricts to an automorphism of the smooth connected commutative affine k-group $U'_t := c_t(U) \bigcap U'$. Clearly $c'_t(U') \subset U'_t$, and by dimension induction we know that c'_t is an automorphism of $c'_t(U')$, so the reduction to the case of commutative U is reduced to the following lemma:

Lemma 3.2. *The inclusion $c'_t(U') \subset U'_t$ of k-groups is an equality.*

Proof. In view of the smoothness and connectedness of these k-groups, it is equivalent to compare their Lie algebras (inside $\mathrm{Lie}(U)$). Let $V = \mathrm{Lie}(U)$, and let $\mathrm{Ad}_U : T \to \mathrm{GL}(V)$ denote the adjoint representation arising from the T-conjugation action on U, so $\mathrm{Lie}(c_t) = \mathrm{Ad}_U(t) - \mathrm{id}$ on V. Note that $\mathrm{Ad}_U(t)$ is a semisimple automorphism, due to the algebraicity of Ad_U. Thus, we may form the eigenspace decomposition $V = \bigoplus V_\lambda$ for $\mathrm{Ad}_U(t)$, so $\mathrm{Lie}(c_t)$ acts as $\lambda - 1$ on each V_λ. It follows that the image of $\mathrm{Lie}(c_t)$ is $\bigoplus_{\lambda \neq 1} V_\lambda$, but a priori this image is merely contained in $\mathrm{Lie}(c_t(U))$. The obstruction to equality here is the problem of surjectivity of the tangent map at the identity e for

$c_t : U \to c_t(U)$. In other words,

$$\mathrm{Tan}_e(c_t(U)) = (\mathrm{Tan}_e(c_t(U)) \bigcap V_1) \oplus (\oplus_{\lambda \neq 1} V_\lambda).$$

By similar reasoning with $V' = \mathrm{Lie}(U')$ and its eigenspaces $V'_\lambda = V' \bigcap V_\lambda$ relative to $\mathrm{Ad}_{U'}(t) = \mathrm{Ad}_U(t)|_{V'}$, the image of $\mathrm{Lie}(c'_t)$ is $\bigoplus_{\lambda \neq 1} V'_\lambda$. But $c'_t : c'_t(U') \to c'_t(U')$ is an isomorphism by the dimension induction, so the image of $\mathrm{Lie}(c'_t)$ coincides with $\mathrm{Lie}(c'_t(U'))$. We thereby get the containment

$$\begin{aligned} \mathrm{Lie}(c'_t(U')) \subset \mathrm{Lie}(U'_t) = \mathrm{Tan}_e(c_t(U)) \bigcap V' &= (\mathrm{Tan}_e(c_t(U)) \bigcap V'_1) \oplus (\oplus_{\lambda \neq 1} V'_\lambda) \\ &= (\mathrm{Tan}_e(c_t(U)) \bigcap V'_1) \oplus \mathrm{Lie}(c'_t(U')). \end{aligned}$$

It therefore remains to show that $\mathrm{Tan}_e(c_t(U)) \bigcap V'_1 = 0$, or more specifically that $c_t : U \to c_t(U)$ is surjective on tangent spaces at the identity. We shall prove that this morphism is *smooth*.

Since t is fixed, rather than work with c_t we may apply left translation by t^{-1} to express the problem in terms of the orbit morphism $U \to G$ defined by $u \mapsto ut^{-1}u^{-1}$. More precisely, $t^{-1}c_t(U)$ is the (reduced) locally closed image of the orbit morphism, and we know from our study of orbit morphisms in the previous course (see 18.1.1, especially its proof) that an orbit of a linear algebraic group under an action on an affine algebraic scheme (such as the U-action on G through conjugation) is identified with the fppf quotient modulo the stabilizer *scheme* of the point through which the orbit is being taken. That is, $t^{-1}c_t(U)$ is identified with the quotient scheme $U/Z_U(t^{-1})$ modulo the U-centralizer of t^{-1}. More specifically, $U \to t^{-1}c_t(U)$ is faithfully flat with fibers that are translates of the stabilizer scheme $Z_U(t^{-1})$, so provided that $Z_U(t^{-1})$ is smooth we can conclude that $c_t : U \to c_t(U)$ is fppf with smooth fibers, so it is a smooth morphism and therefore surjective on tangent spaces at all k-points.

Now the problem is rather more concrete: if H is a linear algebraic subgroup of a linear algebraic group G over a field k (such as $U \subset G$ above) and if $s \in G(k)$ is semisimple (such as t^{-1} above) and normalizes H then we claim that the centralizer scheme $Z_H(s) = H \bigcap Z_G(s)$ is smooth. To prove this, we shall express the problem in terms of the smooth (possibly disconnected!) subgroup M generated by s (i.e., the Zariski closure of $\langle s \rangle$ in G). Note that M normalizes H, and $Z_H(s) = Z_H(M)$. Also, the identity component M^0 is a torus and M/M^0 has finite order not divisible by $\mathrm{char}(k)$. Indeed, by choosing a k-subgroup inclusion $G \hookrightarrow \mathrm{GL}_n$ and extending scalars to $\overline{k}$ so that s diagonalizes, this becomes an elementary assertion about diagonal matrices. Our problem now has nothing at all to do with G; it is a special case of the next lemma. ■

Lemma 3.3. *Let M be a smooth commutative affine group over a field k such that M^0 is a torus and M/M^0 has order not divisible by* $\mathrm{char}(k)$. *For any action by M on a smooth k-scheme Y, the scheme-theoretic fixed locus Y^M is smooth.*

This is proved as Lemma 3.2 in the handout "Applications of Grothendieck's theorem on Borel subgroups". Now we may assume U is commutative, so $c_t : U \to U$ is a homomorphism (as we noted above for U'). Now assume $\mathrm{char}(k) = 0$, so by Lemma 2.3 (and the discussion immediately following it) in the handout "Applications of Grothendieck's theorem on Borel subgroups", the group U is a power of $\mathbf{G}_a$ and upon fixing an isomorphism $U \simeq V := \mathbf{G}_a^n$ the resulting T-action on V is *linear* (i.e., commutes with the diagonal $\mathbf{G}_m$-scaling action on V). Thus, in characteristic 0 we have reduced the general problem to the toy example with linear representations of tori that was solved at the outset in any characteristic.

Now assume $\mathrm{char}(k) = p > 0$. How can we adapt the preceding argument to work? The first problem is to get to the case when U is isomorphic to $\mathbf{G}_a^n$ as k-groups (without dwelling on whether this isomorphism makes the T-action on U appear linear). Even though U is commutative, if it is not p-torsion then there is no such k-group isomorphism (e.g., the group functor W_2 of length-2 Witt vectors over k-algebras is an extension of $\mathbf{G}_a$ by $\mathbf{G}_a$, but W_2 is not p-torsion). But we can pass to the p-torsion case rather easily, as follows. Since U is commutative, the map $c_t : U \to U$ is a *homomorphism*. Hence, to prove the isomorphism property for c_t on $c_t(U)$ it suffices to treat the successive quotients of a T-equivariant composition series for U by smooth connected closed subgroups. The composition series $\{U[p^i]\}$ is characteristic and has p-torsion successive quotients, but the torsion subgroup schemes $U[p^i]$ may fail to be smooth or connected. Instead, we use the characteristic composition series $\{p^i(U)\}$ whose terms are smooth and connected. In this way, we may also arrange that U is p-torsion, so by Lemma B.1.10 in "Pseudo-reductive Groups" it follows that U is a power of $\mathbf{G}_a$.

It remains, with $\mathrm{char}(k) = p > 0$, to relate an abstract semi-direct product $T \ltimes V$ with $V \simeq \mathbf{G}_a^n$ to the special case when the T-action on V appears linear under some k-group isomorphism $V \simeq \mathbf{G}_a^n$. Here again, the main difficulty was solved by Tits: he proved the remarkable result that $V = V^T \times V'$ where V' is a T-stable smooth connected k-subgroup admitting a k-group isomorphism $V' \simeq \mathbf{G}_a^{n'}$ under which the T-action on V' becomes linear! This is Theorem B.4.3 in "Pseudo-reductive Groups". Visibly $c_t(V) = c_t(V')$, so we may replace V with V' to once again reduce to the solved toy case of a linear action. This completes the proof of Theorem 3.1. ■

The following corollary is loosely analogous to the fact that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups.

Corollary 3.4. *Let G be a nilpotent connected solvable linear algebraic k-groups. The maximal k-torus T is unique, and the natural homomorphism $T_{\overline{k}} \times \mathscr{R}_u(G_{\overline{k}}) \to G_{\overline{k}}$ is an isomorphism.*

Proof. By Theorem 3.1, T is central in G and so it is the only maximal k-torus in G (as $T_{\overline{k}}$ is maximal in $G_{\overline{k}}$ and is invariant under $G(\overline{k})$-conjugation). The direct product isomorphism expresses the structure of connected solvable linear algebraic $\overline{k}$-groups in view of the centrality of $T_{\overline{k}}$. ■

It is *not* true over imperfect k, even in the commutative case, that $\mathscr{R}_u(G_{\overline{k}})$ descends to a k-subgroup of G. That is, the direct product description of $G_{\overline{k}}$ in Corollary 3.4 cannot be descended over k when k is imperfect. A counterexample is the Weil restriction $G = \mathrm{R}_{k'/k}(\mathbf{G}_m)$ for a nontrivial finite purely inseparable extension k'/k. This is a group of dimension $[k' : k] > 1$ for which $G(k_s) = {k'_s}^{\times}$ has no nontrivial p-torsion but the evident 1-dimensional k-torus $\mathbf{G}_m \subset G$ is maximal (because $G/\mathbf{G}_m$ is killed by the $[k' : k]$-power map, as may be checked on k_s-points and hence is unipotent). The absence of nontrivial k_s-rational p-power torsion shows that this G cannot contain a nontrivial connected unipotent linear algebraic k-subgroup U (as otherwise $U(k_s)$ would be infinite and p-power torsion inside $G(k_s)$).