1. Motivation

This handout aims to establish in the general case two key features of the split case: the applicability of BN-pair formalism (a.k.a. “Tits systems”) and a theory of relative root groups which permits a refined formulation of the relative Bruhat decomposition (corresponding geometrically to the description of each locally closed Bruhat cell as an explicit product variety of a Borel subgroup against an affine space). We also use root groups to relate $k$-isotropic “$k$-simple factors” to irreducible components of $k\Phi$. We saw with both Weil restriction and special unitary groups in class (see §11.2) that relative root groups can have dimension larger than 1, and that this can occur even in the absolutely simple case.

In the split case, much of this was explained in the handout on the geometric Bruhat decomposition. Let’s recall the main argument concerning the refined Bruhat decomposition in the split case. Let $G$ be a split connected reductive group over a field $k$. Choose a split maximal torus $T$ and a Borel $k$-subgroup $B$. Let $W = N_G(T)(k)/T(k) = W(\Phi(G, T))$. The Bruhat decomposition gives

$$G(k) = \prod_{w \in W} B(k)n_wB(k)$$

where $n_w \in N_G(T)(k)$ is a choice of representative of $w$. There is usually tremendous non-uniqueness in expressions $bn_wb'$ for a given $g \in G(k)$, so for computations and other purposes it is convenient to refine our description of these double cosets.

For any $w \in W$, let $\Phi^+_w = \{ a \in \Phi^+ | w^{-1}.a \in \Phi^+ \}$ and let $\Phi'_w = \Phi^+ - \Phi^+_w$ be its complement. Choose an enumeration of $\Phi^+$ for which $\Phi'_w$ appears before $\Phi^+_w$. Thus,

$$Un_w = \prod_{a \in \Phi'_w} U_a \times \prod_{a \in \Phi^+_w} U_a n_w = (\prod_{a \in \Phi'_w} U_a)n_w \cdot \prod_{a \in \Phi^+_w} (n_w^{-1} U_a n_w).$$

But $n_w^{-1} U_a n_w = U_{w^{-1}.a} \subset U$ for $a \in \Phi^+_w$, so

$$Un_w U = (\prod_{a \in \Phi'_w} U_a)n_w \times U = \prod_{a \in \Phi'_w} U_a \times n_w \prod_{a \in \Phi^+} U_a.$$

The same computation works on $k$-points, and if we include a factor of $T$ (which normalizes $U$ and is normalized by $N_G(T)(k)$) then we arrive at the description

$$B(k)n_wB(k) = \prod_{a \in \Phi'_w} U_a(k) \times T(k)n_w \times \prod_{a \in \Phi^+} U_a(k)$$

via multiplication inside $G(k)$. This really is a direct product decomposition, since we can move $n_w$ all the way to the left to arrive at the equivalent description

$$n_w \cdot \prod_{a \in \Phi'_w} U_{w^{-1}.a}(k) \times T(k) \times \prod_{a \in \Phi^+} U_a(k) \subset n_w(U^- \times T \times U)(k)$$

inside a translate of the open cell associated to the pair $(T, B)$.

In this description, the order of enumeration $\Phi'_w$ might seem to matter in terms of the image of $\prod_{a \in \Phi'_w} U_a(k)$ inside $B(k)$. To explain why it is independent of the enumeration, we shall use a dynamic technique from [CGP] whose proof rests on elementary properties of
root systems: any closed set of roots $\Psi$ in a (possibly non-reduced) root system $\Phi_0$ is the intersection $\Phi_0 \cap A$ of $\Phi_0$ with a subsemigroup $A$ of the ambient $Q$-vector space $V$ generated by $\Phi_0$, so any such $\Psi$ is the set of roots in the subsemigroup of $V$ generated by $\Psi$. This latter condition on a subset $\Psi$ of $\Phi_0$ is called being saturated (so any closed set of roots is saturated).

For any saturated set of roots $\Psi$ in the reduced root system $\Phi$, we will see in Proposition 3.3 that for general $G$ (without a split hypothesis) the associated root groups $U_a$ ($a \in \Psi$) directly span in any order a common unipotent smooth connected $k$-subgroup $U_{\Psi} \subset G$ (normalized by $T$). This rests on a “direct spanning” construction from [CGP, §3.3] that has absolutely nothing to do with the theory of reductive groups.

We can summarize the refined Bruhat decomposition in the split case as

$$B(k)n_w B(k) = U_{\Phi_0}(k) \times T(k) \times U_{\Phi^+}(k),$$

where $U_{\Psi} := \prod_{a \in \Psi} U_a$ (multiplication in any order) for any saturated subset $\Psi$ of $\Phi(G, T)$. As was noted in Remark 1.1 in the handout on the geometric Bruhat decomposition, $\#\Phi_0'$ is the length $\ell(w)$ of $w$ relative to the set of reflections $\{r_a\}_{a \in \Delta}$ in $W(G, T)$ giving a Coxeter presentation, where $\Delta$ is the basis of $\Phi^+$. This gives geometric meaning to the Bruhat cells $Bn_w B = A^{\ell(w)}_k \times n_w B$ in the split case. A version of this in the general case is given on $k$-points in Corollary 3.8 using a direct spanning property on $k$-points, and it is crucial in many applications (such as the development of Bruhat–Tits theory).

2. Root groups and Tits systems

Now let $G$ be an arbitrary connected reductive group over a field $k$, and let $S$ be a maximal $k$-split torus. Let $S_0$ be the maximal $k$-split torus and $S' = (S \cap G')^0_{\text{red}}$ (where $G' := \mathcal{D}(G)$), so $S_0 \times S' \to S$ is an isogeny and under the decomposition $X(S)_Q = X(S_0)_Q \oplus X(S')_Q$. $\Phi \cap k\Phi = \Phi(G, S)$ spans $X(S')_Q$ making $(X(S')_Q, k\Phi)$ a root system. For the rest of the handout we may as well assume $k\Phi \neq \emptyset$, which is to say $S$ is not central in $G$, as otherwise there is no content to anything that follows.

We begin by defining the root group $U_a \subset G$ associated to each $a \in \Phi$. Beware that in general (unlike in the split case) this may have rather large dimension, and can even be non-commutative; we will discuss the precise structure of such $U_a$ later. The crux of the matter is the following general dynamical construction that has nothing to do with reductive groups and was recorded earlier in the course when we prepared for the discussion of subgroups directly spanned by root groups in the split case.

**Proposition 2.1.** Let $G$ be a smooth connected affine $k$-group equipped with an action by a split $k$-torus $S$, and let $A \subseteq X(S)$ be a subsemigroup not containing $0$. There exists a unique $S$-stable smooth connected $k$-subgroup $U_A(G)$ such that $\text{Lie}(U_A(G))$ is the span of the $a$-weight spaces for all $a \in A \cap \Phi(G, S)$. This $k$-subgroup is unipotent and contains any $S$-stable smooth connected $k$-subgroup $H \subseteq G$ such that all $S$-weights on $\text{Lie}(H)$ occur in $A$.

Additional properties of $U_A(G)$ will be reviewed as the need arises below.

**Proof.** This is [CGP, Prop. 3.3.6], where the notation $H_A(G)$ is used (as it is not assumed there that $0 \not\in A$, so $H_A(G)$ might not be unipotent). ■
For any \( a \in X(S) - \{0\} \), let \( \langle a \rangle \) be the semigroup consisting of positive integral multiples of \( a \). For \( a \in \k \Phi \), \( U_a := U_{\langle a \rangle}(G) \) is the smooth connected unipotent \( k \)-subgroup in \( G \) that is normalized by \( S \) and for which \( \text{Lie}(U_a) \) is the span of the weight spaces in \( \text{Lie}(G) \) for the weights in \( \Phi(G,S) \) that are positive integral multiples of \( a \). Moreover, \( U_a \) contains every smooth connected \( k \)-subgroup \( H \subseteq G \) normalized by \( S \) such that \( S \) acts on \( \text{Lie}(H) \) only with weights that are positive integral multiples of \( a \). In particular, \( U_{2a} \subseteq U_a \) if \( 2a \in \k \Phi \).

**Definition 2.2.** The \( k \)-group \( U_a \) is called the root group of \( G \) associated to \( a \in \k \Phi \).

We may sometimes refer to such \( U_a \) are relative root groups for emphasis (to distinguish from the split case). Here is a more concrete description of \( U_a \). Let \( \mathcal{R}_a = \ker a \) (which might not be smooth if \( a \) is divisible in \( X(S) \), and can happen when \( \k \Phi \) has irreducible factors of type \( C_n \) or type \( BC_n \) with \( n > 1 \)) and let \( G_a = Z_G(\mathcal{R}_a)^0 \). By the handout “Reductive centralizer”, \( G_a \) is a connected reductive \( k \)-subgroup of \( G \) (visibly containing \( S \)), and \( \Phi(G_a,S) \) has rank 1: it consists of the roots that are integer multiples of \( a \) (since we used the centralizer of the entire kernel of \( a \), not just \( (\ker a)^0 \)).

**Lemma 2.3.** As \( k \)-subgroups of \( G \), we have \( U_a \subset G_a \).

*Proof.* The smooth connected root group \( U_a \) is normalized by \( S \). We have to show that under the resulting \( S \)-action, the closed \( k \)-subgroup \( \ker a \) acts trivially on \( U_a \). Its effect on the span \( \text{Lie}(U_a) \) of the weight spaces inside \( \mathfrak{g} \) for the positive integral multiples of \( a \) in \( \k \Phi \) is certainly trivial.

But rather generally, the action on a smooth connected affine group by a multiplicative type group is trivial when the action on the Lie algebra is trivial! This follows from a calculation on infinitesimal neighborhoods of the identity, using the complete reducibility of linear representations of multiplicative type groups. A proof is given in the self-contained second paragraph of the proof of [CGP, Cor. A.8.11].

For \( \lambda_a \in X_a(S) \) such that \( \langle a, \lambda_a \rangle > 0 \), we have \( U_{G_a}(\lambda_a) \subseteq U_{U_a}(\lambda_a) = U_a \) inside \( G_a \). This inclusion is between smooth connected \( k \)-groups, and it is an equality on Lie algebras (as \( \Phi(G_a,S) \) consists of the integer multiples of \( a \) inside \( \k \Phi \), and \( \text{Lie}(U_a) \) is the span of the positive integral multiples of \( a \) in \( \k \Phi \)). Thus, we get an equality as \( k \)-subgroups: \( U_a = U_{G_a}(\lambda_a) \).

Now we obtain two nice consequences.

**Proposition 2.4.** If \( H \subset G \) is a closed \( k \)-subgroup scheme normalized by \( S \) then \( U_a \cap H = U_H(\lambda_a) \). In particular, if \( H \) is smooth and connected then so is \( U_a \cap H \).

Since \( P_{G_a}(\pm \lambda_a) \) are the two proper parabolic \( k \)-subgroups of \( G_a \), we recover an alternative description of \( U_a \):

**Proposition 2.5.** For the unique proper parabolic \( k \)-subgroup \( P_a \subset G_a = Z_G(\ker a)^0 \) that contains \( S \) and has \( a \) as an \( S \)-weight on its Lie algebra, \( U_a = \mathbb{R}_{a,\k}(P_a) \).

Since \( P_{\pm a} \) are the only two proper parabolic \( k \)-subgroups of \( G_a \) containing \( S \), so each is minimal, they each contain \( Z_{G_a}(S) = Z_G(S) \) as a Levi factor. In particular, \( U_a \) is normalized by \( Z_G(S) \) for any \( a \in \k \Phi \). The dynamic description \( U_a = U_{G_a}(\lambda_a) \) also implies that \( U_a \) is a \( k \)-split smooth connected unipotent group (a general property of the \( U(\lambda) \) construction
[CGP, Prop. 2.1.10]). When $G$ is split, so $k\Phi$ is a reduced root system, we recover the notion of root group in the split theory.

The commutativity and vector group properties of relative root groups will be explored later. For now, we use root groups to establish a link with the notion of Tits system. Recall from the previous course:

\textbf{Definition 2.6.} A Tits system (or BN-pair) is a 4-tuple $(\mathcal{G}, B, N, \Sigma)$ where $\mathcal{G}$ is an abstract group, $B$ and $N$ are subgroups, and $\Sigma \subseteq N/(B \cap N)$ is a subset such that the following four axioms are satisfied:

- (T1) $B \cup N$ generates $\mathcal{G}$ and $B \cap N$ is normal in $N$,
- (T2) the elements of $\Sigma$ have order 2 in the quotient $W := N/(B \cap N)$ and generate $W$,
- (T3) for all $\sigma \in \Sigma$ and $w \in W$, $\sigma Bw \subseteq BwB \cup B\sigma wB$ (using any representatives for $\sigma$ and $w$ in $N$, the choices of which do not matter),
- (T4) $\sigma B \sigma \not\subseteq B$ for all $\sigma \in \Sigma$ (which is equivalent to $\sigma B \sigma \neq B$, since $\sigma^2 = 1$ in $W$).

We refer the reader to §2 in Chapter IV of Bourbaki for the basic properties of Tits systems. By Remark (1) in §2.5 of Chapter IV of Bourbaki, $\Sigma$ is uniquely determined by $(\mathcal{G}, B, N)$ (thereby explaining why such data is often called a BN-pair for $\mathcal{G}$).

\textbf{Theorem 2.7} (Borel–Tits). Let $N = N_G(S)$, $Z = Z_G(S)$, and $P$ a minimal parabolic $k$-subgroup of $G$ containing $S$. Let $k\Delta$ be the basis of the positive system of roots $k\Phi^+ = \Phi(P, S)$, and let $R = \{r_a \mid a \in k\Delta\}$ be the associated set of simple positive reflections. The 4-tuple $(G(k), P(k), N(k), R)$ is a Tits system with Weyl group $kW (= N(k)/Z(k))$.

This is the standard Tits system associated to $(G, S, P)$. Since $P(k) \cap N(k) = Z(k)$, we have $N(k)/(P(k) \cap N(k)) = : kW = W(k\Phi)$. That is, the Weyl group of the standard Tits system naturally coincides with $W(G, S)$.

\textit{Proof.} The relative Bruhat decomposition $G(k) = \coprod_{w \in kW} P(k)n_a P(k)$ implies axiom (T1). Moreover, the quotient group $N(k)/Z(k) = kW = W(k\Phi)$ is generated by $R$. This is axiom (T2). To prove axiom (T4), for $r = r_a \in R$ (with $a \in k\Delta \subset k\Phi$) it suffices to prove that $rP(k)r \neq P(k)$. Clearly $rP$ contains $rUr_a = U_{r(a)} = U_{-a}$, so it suffices to prove that $U_{-a}(k)$ is not contained in $P(k)$. We will prove $U_{-a} \cap P = 1$, so $U_{-a}(k) \cap P(k) = 1$, which does the job since $U_{-a}(k) \neq 1$ (as the unipotent smooth connected $k$-group $U_{-a}$ is nontrivial and $k$-split).

By Proposition 2.4, $U_{-a} \cap P$ is smooth and connected. But its Lie algebra is trivial since $\Phi(P, S) = k\Phi^+$ does not contain any $S$-weights that occur on $U_{-a}$, so $U_{-a} \cap P = 1$ as desired.

It remains to prove axiom (T3). That is, for $r := r_a$ (with $a \in k\Delta$) and $w \in kW$ we claim that

\[ rP(k)\{w, rw\}P(k) \subseteq P(k)\{w, rw\}P(k). \]

This is given in the proof of 21.15 in the 2nd edition of Borel’s textbook on linear algebraic groups, and here we give a mild reformulation of that calculation.

Let $\lambda : GL_1 \to S$ be a 1-parameter $k$-subgroup such that $P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$, so the minimality of $P$ implies that $Z_G(\lambda) = Z_G(S) = Z$ and $U_G(\lambda) = \mathcal{R}_{u,k}(P)$. Thus, $P = Z \ltimes \mathcal{R}_{u,k}(P)$ and $k\Phi^+ = \Phi(P, S) = (k\Phi)_{\lambda > 0} = (k\Phi)_{\lambda > 0}$, so $\langle a, \lambda \rangle > 0$. More generally, a
k-root $b \in k\Phi$ is positive (i.e., it lies in $k\Phi^+$) if and only if $\langle b, \lambda \rangle > 0$. If necessary, replacing $w$ with $rw$, we may (and we will) assume that $w^{-1}(a)$ is positive.

Now let $U = R_{a,k}(P) = U_G(\lambda)$, so for $G_a := Z_G(\ker a)^0$ the root group $U_a$ is equal to $U_{G_a}(\lambda) = G_a \cap U$. Let $S_a = (\ker a)^{\text{red}}$. As $a$ is nondivisible in $k\Phi$ (since $a \in k\Delta$!), an element of $k\Phi$ is a rational multiple of $a$ if and only if it is an integral multiple of $a$. Thus, the containment $G_a \subseteq Z_G(S_a)$ between smooth connected $k$-groups is an equality via comparison of their Lie algebras.

Choose a 1-parameter $k$-subgroup $\lambda_a : GL_1 \to S_a$ such that $\langle b, \lambda_a \rangle > 0$ for all $b \in k\Delta - \{a\}$. Note that $\langle a, \lambda_a \rangle = 0$, so $G_a = Z_G(\lambda_a)$. Let $V = U_G(\lambda_a)$, so $P_G(\lambda_a) = Z_G(\lambda_a) \rtimes U_G(\lambda_a) = G_a \ltimes V$ and hence $U = U_a \ltimes V$. We also noted above that $P = Z \ltimes U$. Since $G_a$ contains $Z$ as well as a representative of $r = r_a$ (by definition of $r_a$), it follows that $V$ is normalized by $r$ and hence,

$$r P(k) \{w, rw\} P(k) = Z(k) r U(k) \{w, rw\} P(k) = Z(k) V(k) r U_a(k) \{w, rw\} P(k).$$

Thus,

$$r P(k) \{w, rw\} P(k) \subseteq P(k) G_a(k) w P(k).$$

For the parabolic $k$-subgroup $P_a := P_G(\lambda) = G_a \cap P = Z \ltimes U_a$ of the connected reductive $G_a$, the Bruhat decomposition for $G_a(k)$ with respect to $P_a(k)$ implies that

$$G_a(k) = Z(k) U_a(k) \{1, r\} U_a(k),$$

so

$$r P(k) \{w, rw\} P(k) \subseteq P(k) \{1, r\} U_a(k) w P(k).$$

But the positivity of $w^{-1}(a)$ implies that $w^{-1}U_a w \subseteq P$, so

$$r P(k) \{w, rw\} P(k) \subseteq P(k) \{w, rw\} P(k).$$

\[\square\]

Remark 2.8. Since $kW = W(k\Phi)$ acts simply transitively on the set of minimal parabolic $k$-subgroups of $G$ containing $S$ (denote this action as $(w, P) \mapsto w \circ P$), if $P$ is such a $k$-subgroup corresponding to a positive system of roots $\pm \lambda^+ \Phi$ then for a suitable $w_0 \in kW$ the $k$-subgroup $w_0 \circ P$ corresponds to the positive system of roots $\pm \lambda^+ \Phi$. Writing $P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ for some $\lambda \in X_*(S)$, we have $w_0 \circ P = P_G(-\lambda)$. Thus, $P \cap w_0 \circ P = Z_G(\lambda) = Z_G(S) = Z$, so $Z(k) \subseteq \bigcap_{w \in kW} w P(k) w^{-1} \subseteq P(k) \cap w_0 P(k) w_0^{-1} = Z(k)$; i.e., $Z(k) = \bigcap_{w \in kW} w P(k) w^{-1}$.

For $N := N_G(S)$ we have $Z(k) = P(k) \cap N(k)$ due to the minimality of $P$, so

$$\bigcap_{n \in N(k)} n P(k)n^{-1} = P(k) \cap N(k).$$

This says that the Tits system in Theorem 2.7 is saturated in the sense of Exercise 5(a) in §2 of Chapter IV in Bourbaki. The established property that $P(k)$ is a semidirect product of its subgroup $Z(k) = P(k) \cap N(k)$ against a nilpotent subgroup $U_G(\lambda)(k)$ is expressed by saying that the Tits system is split.

In §2 of the handout on Tits systems, we saw that the connection to Tits systems in the split case yields striking simplicity results for $G(k)/Z_G(k)$ in the split simply connected case. That simplicity proof relied on an equality \textquoteright\textquoteright $G(k) = G(k)^+$\textquoteright\textquoteright which says that $G(k)$ is
generated by the rational points of the split $k$-unipotent radicals of parabolic $k$-subgroups of $G$ (or equivalently, just for the minimal parabolic $k$-subgroups, since if $Q \subset P$ is a containment of parabolic $k$-subgroups then $\mathcal{R}_{u,k}(P) \subset \mathcal{R}_{u,k}(Q)$, as we may check over $\overline{k}$). This property in the split case is not true in general, and its validity over global fields lies quite deep (see §3 in the handout on Tits systems). Consequently, we do not discuss simplicity results for $G(k)/Z_G(k)$ beyond the split case, and instead demonstrate the power of the link with Tits systems via the following striking result.

**Theorem 2.9.** With notation as in Theorem 2.7, the following hold:

1. There are $2^{|R|}$ subgroups of $G(k)$ containing $P(k)$, and any such subgroup equals $Q(k)$ for a unique parabolic $k$-subgroup $Q$ containing $P$.
2. For parabolic $k$-subgroups $Q$ and $Q'$ of $G$, $Q \subset Q'$ if and only if $Q(k) \subset Q'(k)$.

Informally, this theorem says that “parabolic subgroups” of $G(k)$ relative to its Tits system structure (see §2.6 in Chapter IV of Bourbaki for this terminology) are in inclusion-preserving bijective correspondence with the parabolic $k$-subgroups of $G$ via the formation of rational points. Note that we allow the case that $k$ is finite, and in (ii) we don’t assume $Q$ and $Q'$ share a common split maximal $k$-torus.

**Proof.** Since $(G(k), P(k), N(k), R)$ is a Tits system (with Weyl group $kW$), by Theorem 3(b) in §2.5 of Chapter IV in Bourbaki there is a natural bijection $X \mapsto G(k)_X$ from the set of subsets $X \subseteq R$ to the set of subgroups of $G(k)$ containing $P(k)$, so there are exactly $2^{|R|}$ subgroups of $G(k)$ containing $P(k)$.

Explicitly, in terms of the relative Bruhat decomposition $G(k) = \bigsqcup_{w \in kW} P(k) h_w P(k)$ for a choice of representative $h_w \in N(k)$ of each $w \in kW$, $G(k)_X$ is the union of the $P(k)$-double cosets indexed by the elements of the subgroup $kW_X$ of $kW$ generated by $X$. Note that the subgroup $G(k)_X$ is the unique subgroup of $G(k)$ containing $P(k)$ such that the image of $N(k) \cap G(k)_X$ in $kW$ is $kW_X$. (In particular, $kW_X$ determines $X$.) In general $kW_X \cap kW_{X'} = kW_{X \cap X'}$, so $G(k)_X \cap G(k)_{X'} = G(k)_{X \cap X'}$. It follows that if $G(k)_X \subseteq G(k)_{X'}$, then $G(k)_X = G(k)_{X \cap X'}$ and hence $X = X \cap X' \subseteq X'$.

Any smooth closed $k$-subgroup $Q$ of $G$ containing $P$ is parabolic, and there are exactly $2^{|R|}$ such $k$-subgroups (corresponding to subsets $I$ of $k\Delta$). We shall now display them all. For each subset $I \subseteq k\Delta$, we fix a cocharacter $\lambda_I \in X_*(S)$ such that $\langle a, \lambda_I \rangle \geq 0$ for all $a \in k\Delta$, with equality if and only if $a \in I$. Define $P_I = P_G(\lambda_I) = Z_G(\lambda_I)U_G(\lambda_I)$. In view of how $k\Delta$ is defined in terms of $P$, by weight space computations in $g$ we see that $P = P_\emptyset$ and $P_I \subseteq P_J$ when $J \subseteq J'$; in particular, $P \subseteq P_I$. We will prove that if $J \neq J'$ then $P_J(k) \neq P_J'(k)$ (so $P_I \neq P_{I'}$). It would then follow that the collection $\{P_I \mid I \subseteq k\Delta\}$ consists of $2^{|k\Delta|} = 2^{|R|}$ parabolic $k$-subgroups of $G$ containing $P$, and the collection $\{P_I(k) \mid I \subseteq k\Delta\}$ consists of $2^{|R|}$ subgroups of $G(k)$ containing $P(k)$. Hence, the former would be the collection of all parabolic $k$-subgroups of $G$ containing $P$ and the latter would be the collection of all subgroups of $G(k)$ containing $P(k)$, so (i) would be proved.

Fix an $I$, and let $N_I = N \cap P_I = N_{P_I(S)}$. We have noted above that $I$ is determined by the subgroup $kW_I \subseteq kW$ generated by $\{r_a \mid a \in I\} \subseteq R$, so it suffices to show that the image of $N(k) \cap P_I(k) = N_I(k)$ in $kW$ is $kW_I$ (and then $P_I = G(k)_I$, due to the unique
characterization of $G(k)_X$ noted above). This image $N_I(k)/Z(k)$ is the “relative Weyl group” $N_{P_I}(S)(k)/Z_{P_I}(S)(k)$ of $(P_I, S)$, though $P_I$ isn’t reductive for $I \neq k \Delta$.

Let’s first show that the subgroup $W(Z_G(\lambda_I), S)(k) \subseteq kW$ is generated by $\{r_a\}_{a \in I}$ (i.e., it coincides with $kW_I$). Due to how $\lambda_I$ was defined (with $I \subset k \Delta$), the root system $k\Phi_I := \Phi(Z_G(\lambda_I), S)$ is the span of $I$ inside $k\Phi$, with $I$ moreover as a basis. Thus, $W(Z_G(\lambda_I), S)(k)$ is generated by reflections $\{r'_a\}_{a \in I}$. The natural inclusion $W(Z_G(\lambda_I), S)(k) \hookrightarrow kW$ inside $\text{GL}(X(S)_k)$ carries $r'_a$ to $r_a$ for each $a \in I$, due to the well-known fact that for any nonzero finite-dimensional vector space $V$ over a field of characteristic 0 and any finite subgroup $F \subset \text{GL}(V)$, a reflection $r \in F$ is uniquely determined by the line that it negates.

We have shown that $kW_I \subset N_I(k)/Z(k)$ via the inclusion of $Z_G(\lambda_I)$ into $P_I(k) = P_I$, and to prove equality it suffices to show that these finite groups have the same size. More specifically, using the natural identification of $Z_G(\lambda_I)$ with the maximal reductive quotient $\overline{P_I} = P_I(k)/U_G(\lambda_I)$ of $P_I$ and the isomorphism of $S$ onto its maximal $k$-split torus image $\overline{S}$ in $\overline{P_I}$, to complete the proof of (i) it suffices to show that the natural map $N_I(k)/Z(k) \to W(\overline{P_I}, \overline{S})(k)$ is an isomorphism. The key point is that the inclusion $N_I(k)/Z(k) \to W(P_I, S)(k) := (N_{P_I}(S)/Z_{P_I}(S))(k)$ is an equality. This is an analogue for $P_I$ of what we know for $G$, and the proof of that result for connected reductive groups was entirely formal up to the known fact that all maximal $k$-split tori of $P_I$ are $P_I(k)$-conjugate (as $P_I$ is a parabolic $k$-subgroup of the connected reductive $G$) and the following two facts:

**Lemma 2.10.** For any minimal parabolic $k$-subgroup $P$ of $P_I$ containing $S$, $N_{P_I}(S) \cap P = Z_{P_I}(S)$. Moreover, the minimal parabolic $k$-subgroups of $P_I$ are $P_I(k)$-conjugate.

**Proof.** The first assertion is inherited from the same for $G$ in place of $P_I$ (as a parabolic $k$-subgroup of $P_I$ is precisely a parabolic $k$-subgroup of $G$ contained in $P_I$, due to the parabolicity of $P_I$). For the second assertion, note that any parabolic $k$-subgroup of $P_I$ must contain the normal split unipotent smooth connected $k$-subgroup $U_G(\lambda_I)$. Thus, we get an inclusion-preserving bijection between the sets of parabolic $k$-subgroups of $P_I$ containing $S$ and those of the connected reductive $\overline{P_I} = P_I/U_G(\lambda_I)$ containing the maximal $k$-split torus image of $S$. Hence, the known $k$-rational conjugacy result for the reductive $\overline{P_I}$ implies the same for $P_I$ because $P_I(k) \to \overline{P_I}(k)$ is surjective (as $U_G(\lambda_I)$ is split unipotent).

We may now recast our problem in terms of finite étale $k$-groups rather than groups of $k$-points: we want $N_{I'/Z} \to W(\overline{P_I}, \overline{S})$ to be an isomorphism. This is a special case of a general fact having nothing to do with reductive groups:

**Lemma 2.11.** Let $H$ be a smooth connected affine group over a field $k$, $U$ a normal unipotent smooth connected $k$-subgroup, and $H' = H/U$. Let $T$ be a $k$-torus in $H$ and let $T'$ be its image in $H'$. The natural map $N_H(T)/Z_H(T) \to N_{H'}(T')/Z_{H'}(T')$ between finite étale $k$-groups is an isomorphism.

**Proof.** This is [CGP, Lemma 3.2.1] (which permits $U = \text{ker}(H \to H')$ to be a bit more general, but whose proof is self-contained relative to our situation).

To prove assertion (ii) in Theorem 2.9, consider arbitrary parabolic $k$-subgroups $Q$ and $Q'$ of $G$ such that $Q(k) \subseteq Q'(k)$. Fix a minimal parabolic $k$-subgroup $P$ contained in $Q$. By the conjugacy of minimal parabolic $k$-subgroups of $G$, there exists $g \in G(k)$ such that
unipotent radicals of the parabolic $k$ for some $\lambda$

Remark 3.2

Lemma 3.3.8 \[\text{(a general result on smooth connected commutative unipotent groups equipped} \]

The centralizer $Z_G(\ker a)$ has reductive identity component and contains $U_{\pm a}$. Since $Z_G(S) \subseteq Z_G(\ker a)^0$, so we may (and do) replace $G$ with $Z_G(\ker a)^0$. Now $k\Delta = \{a\}$, the Weyl group $kW$ is a group of order 2 generated by $r_a$, and $P := Z \ltimes U_{-a}$ is a minimal parabolic $k$-subgroup of $G$.

Since $k\Phi$ is a root system, the only positive integral multiple of $a$, other than $a$, which can be in $k\Phi$ is $2a$. By construction, $U_a = U_{(a)}(G)$ and $U_{2a} = U_{(2a)}(G)$, so general dynamic principles (see [CGP, Prop. 3.3.5]) ensure that if $2a \in k\Phi$ then $(U_a, U_a) \subseteq U_{2a}, U_{2a} \subseteq Z_{U_a}$, and $U_a/U_{2a}$ is a vector group admitting a unique linear structure relative to which the $S$-action is linear. In such multipliable cases, the underlying $k$-scheme of $U_a$ is isomorphic to an affine space.

Proof. The centralizer $Z_G(\ker a)$ has reductive identity component and contains $U_{\pm a}$. Since $Z_G(S) \subseteq Z_G(\ker a)^0$, so we may (and do) replace $G$ with $Z_G(\ker a)^0$. Now $k\Delta = \{a\}$, the Weyl group $kW$ is a group of order 2 generated by $r_a$, and $P := Z \ltimes U_{-a}$ is a minimal parabolic $k$-subgroup of $G$.

Since $k\Phi$ is a root system, the only positive integral multiple of $a$, other than $a$, which can be in $k\Phi$ is $2a$. By construction, $U_a = U_{(a)}(G)$ and $U_{2a} = U_{(2a)}(G)$, so general dynamic principles (see [CGP, Prop. 3.3.5]) ensure that if $2a \in k\Phi$ then $(U_a, U_a) \subseteq U_{2a}, U_{2a} \subseteq Z_{U_a}$; likewise, if $2a \not\in k\Phi$ then we similarly conclude that $U_a$ is commutative. It then follows from [CGP, Lemma 3.3.8] (a general result on smooth connected commutative unipotent groups equipped with a sufficiently nontrivial torus action, having nothing to do with reductive groups) that $U_a$ is a vector group in the non-multipliable case and that $U_a/U_{2a}$ is a vector group in the multipliable case, with these vector groups admitting a unique $S$-equivariant linear structure.

In particular, in the multipliable case $U_a$ is a $U_{2a}$-torsor over the affine space $U_a/U_{2a}$. Since $U_a$ is a vector group it follows (from the triviality of étale $G_a$-torsors over affine schemes) that $U_a \to U_a/U_{2a}$ admits a section and hence the $k$-scheme $U_a$ is isomorphic to an affine space.

Remark 3.2. Let $G$ be a smooth connected affine $k$-group, $S$ a maximal $k$-split torus of $G$, and $P$ a minimal parabolic $k$-subgroup of $G$ containing $S$. Let $N = N_G(S)$ and $Z = Z_G(S)$. By minimality of $P$ we have $P = P_G(\lambda) = U_G(\lambda) \ltimes Z_G(\lambda)$ with $Z_G(\lambda) = Z$ and $\mathcal{R}_{a,k}(P) = U_G(\lambda)$ for some $\lambda \in X_S(S)$.

Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the $k$-rational points of the $k$-split unipotent radicals of the parabolic $k$-subgroups of $G$. It is immediate from the preceding
proposition that $G(k)^+ \cap N(k)$ maps onto the $k$-Weyl group $kW = N(k)/Z(k)$. Now let $\mathcal{G}$ be a subgroup of $G(k)$ that contains $G(k)^+$, $\mathcal{N} := N(k) \cap \mathcal{G}$, and $\mathcal{B} := P(k) \cap \mathcal{G}$. Then $\mathcal{N}$ maps onto $kW$. As $U_G(\lambda)(k) \subset \mathcal{B}$, we see that $P(k) = U_G(\lambda)(k) \times Z(k) = \mathcal{B} Z(k)$. Moreover, for all $n \in \mathcal{N}$ we have $P(k)nP(k) = \mathcal{B} n \mathcal{B} Z(k)$ and $\mathcal{G} \cap (P(k)nP(k)) = \mathcal{B} n \mathcal{B}$. It is seen from this, using the relative Bruhat decomposition of $G(k)$, that $G(k) = \mathcal{G} \cdot Z(k)$.

Now it is easily deduced from Theorem 2.7 and Remark 2.8 that the 4-tuple $(\mathcal{G}, \mathcal{B}, \mathcal{N}, R)$, with $R$ as in Theorem 2.7, is a Tits system with Weyl group $kW$. We shall call this Tits system a standard Tits system in $\mathcal{G}$. Its rank (i.e., the cardinality of $R$) is equal to the $k$-rank of $\mathcal{D}(G)$.

The equality $G(k) = \mathcal{G} \cdot Z(k)$ implies that all maximal $k$-split tori (resp., all minimal parabolic $k$-subgroups) of $G$ are conjugate to each other under $\mathcal{G}$.

Recall that a subset $\Psi$ of $\Phi(G, S)$ is saturated if the subsemigroup $A$ of $X(S)$ spanned by $\Psi$ does not contain 0 and $\Psi = A \cap \Phi$. By [CGP, Prop. 2.2.7], for any positive system of roots $k\Phi^+$ in $\Phi$, any closed set of roots $\Psi \subseteq k\Phi^+$ is saturated. The two most interesting examples of saturated (even closed) subsets of $k\Phi$ for our purposes are obtained as follows. For any 1-parameter $k$-subgroup $\lambda : \text{GL}_1 \to S$ we can take $\Psi$ to be $k\Phi_{\Lambda > 0}$, and for any linearly independent $a, b \in k\Phi$ we can take $\Psi$ to be the set $(a, b)$ of elements in $k\Phi$ of the form $ma + nb$ for positive integers $m$ and $n$.

We seek to construct a smooth connected unipotent $k$-subgroup $U_\Psi$ attached to any saturated $\Psi$, and to describe it as a direct span (in any order!) of suitable root groups. However, if $k\Phi$ is not reduced and $c, 2c \in \Psi$ then we have to be careful not to use both $U_c$ and $U_{2c}$ in such a direct spanning (as $U_{2c} \subset U_c$).

**Proposition 3.3.** Let $(G, S, k\Phi)$ be as above.

1. For any saturated subset $\Psi$ of $k\Phi$ there is a unique $S$-stable smooth connected unipotent $k$-subgroup $U_\Psi$ such that $\text{Lie}(U_\Psi)$ is the span of the subspaces $\text{Lie}(U_c)$ for $c \in \Psi$. It is normalized by $Z_G(S)$, and directly spanned in any order by the subgroups $U_c$ for $c \in \Psi$ that are not divisible in $\Psi$.

2. For any parabolic $k$-subgroup $P$ of $G$ containing $S$, the subset $\Psi$ of $k\Phi$ consisting of $a \in k\Phi$ such that $-a \notin \Phi(P, S)$ is saturated, and $\mathcal{R}_{a,k}(P) = U_\Psi$.

We will make the description of $\Psi$ and $U_\Psi$ in (2) more explicit in Remark 3.4.

**Proof.** To prove assertion (1), let $A$ be the subsemigroup of $X(S)$ generated by $\Psi$. Then as $\Psi$ is saturated, $0 \notin A$ and $A \cap \Phi(G, S) = \Psi$. Let $U_\Psi = U_A(G)$ as in Proposition 2.1, which ensures (since $0 \notin A$) that $U_\Psi$ is a smooth connected unipotent $k$-subgroup. To prove that $Z_G(S)$ normalizes $U_\Psi$, we note that the description of $U_A(G)$ makes sense without reference to the maximality of $S$ as a $k$-split torus in $G$, and is compatible with any extension of the ground field. Thus, using scalar extension to $k_s$ implies that $(U_\Psi)_{k_s}$ is normalized by $Z_G(S)(k_s)$, and hence $U_\Psi$ is normalized by $Z_G(S)$. The rest of assertion (1) follows easily from [CGP, Theorem 3.3.11] applied to the smooth connected solvable $k$-group $U_\Psi = U_A(G)$ equipped with its natural action by $S$ (using the disjoint union decomposition $\Psi = \bigsqcup \Psi_i$ where the $\Psi_i$ are the non-empty intersections of $\Psi$ with half-lines in $X(S)_Q$).

To prove assertion (2), we fix a $\lambda \in X_*(S)$ such that $P = P_G(\lambda)$. Thus, $\mathcal{R}_{a,k}(P) = U_G(\lambda)$. But $U_\Psi = U_G(\lambda)$ for $\Psi = k\Phi_{\Lambda > 0}$ by uniqueness in (1), so (2) is proved.
Remark 3.4. Fix a positive system of roots $\kappa \Phi^+$, with $\kappa \Delta$ its basis, so we get the notion of standard parabolic $k$-subgroup of $G$ as those parabolic $k$-subgroups $Q$ containing $S$ such that $k \Phi^+ \subset \Phi(Q, S)$, or equivalently $Q$ contains the minimal parabolic $k$-subgroup $P_0 \supset S$ such that $\Phi(P_0, S) = k \Phi^+$. We know that the standard parabolic $k$-subgroups are in inclusion-preserving bijection with the subsets $I \subset k \Delta$ via the condition $\Phi(k P_I, S) = k \Phi^+ \cup [I]$. As a refinement of Proposition 3.3(2), we shall now describe $\mathcal{R}_{u,k}(k P_I)$ and a Levi factor directly in terms of $I$.

The closed set $\Psi_I$ of roots occurring as $S$-weights on $\text{Lie}(\mathcal{R}_{u,k}(k P_I))$ are precisely the positive relative roots not in $[I]$, which is to say those whose $\kappa \Delta$-expansion is not supported inside $I$. Thus, $\mathcal{R}_{u,k}(k P_I)$ is directly spanned in any order by the relative root groups $U_c$ for non-divisible $c \in k \Phi^+$ not contained in $[I]$. A Levi factor $L_I$ of $k P_I$ containing $S$ must contain contains $Z_G(S)$ and must contain the root groups $U_c$ for $c \in [I]$, so $L_I$ is generated by those $k$-subgroups for Lie algebra reasons (in view of our description of $\mathcal{R}_{u,k}(k P_I)$); in particular, such an $L_I$ containing $S$ is uniquely determined.

But we can describe $L_I$ in the form $Z_G(S_I)$ for an explicit subtorus $S_I \subset S$ as follows. Let

$$S_I = (\bigcap_{a \in I} \ker a)^{\text{red}}_0 \subset S.$$  

Note that for every $c \in [I]$ we have $c|_{S_I} = 1$, so the $S_I$-centralizer in $U_c$ has full Lie algebra. But such a torus centralizer in $U_c$ is smooth and connected, so $S_I$ centralizes $U_c$. (This argument permits the possibility that $c$ is multipliable, in which case $U_c$ is generally non-commutative!) In other words, the smooth connected $k$-subgroup $Z_G(S_I) \subset G$ contains $L_I$. We claim that the inclusion $L_I \subset Z_G(S_I)$ is an equality, thereby completing our explicit description

$$P_I = Z_G(S_I) \ltimes U_{\Psi_I}$$

in terms of $I$.

Since $Z_G(S_I)$ is smooth and connected, it remains to check that the only nontrivial $S$-weights occurring on its Lie algebra are the ones in $[I]$. That is, we claim that if $a \in k \Phi$ and $a|_{S_I} = 1$ then $a \in [I]$ (the converse being obvious). Since $I$ is a linearly independent subset of $X(S)_\mathbb{Q}$, the characters of $S$ that are trivial on $S_I$ are exactly those that are a $\mathbb{Q}$-linear combination of elements of $I$. But $\kappa \Delta$ is a basis of $k \Phi$ containing $I$, so a relative root is in the $\mathbb{Q}$-span of $I$ if and only if it is in the $\mathbb{Z}$-span of $I$ (i.e., lies in $[I]$).

For independent $a, b \in k \Phi$, if we apply Proposition 3.3(1) to the saturated subset $\Psi = (a, b)$ then the construction of $U_{\Psi}$ and [CGP, Prop. 3.3.5] yield:

**Corollary 3.5.** We have

$$(U_a, U_b) \subseteq U_{(a, b)} = \prod_i U_{c_i}$$

where the direct spanning is taken with respect to an arbitrary enumeration $\{c_i\}$ of the set of non-divisible elements of $(a, b)$.

**Remark 3.6.** For $a \in k \Phi$, let $M_a = m(u) Z_G(S)$, where $u$ is any nontrivial element of $U_a(k)$ and the element $m(u) \in N_G(S)(k)$ is as in Lemma 4.2 of the handout on Cartan’s connectedness theorem. It is easy to see using Proposition 3.1, Corollary 3.5, and Lemma 4.2
from the handout on Cartan’s connectedness theorem that \( (Z_G(S)(k), (U_a(k), M_a(k)))_{a \in \Phi} \) is a generating root datum (donnée radicielle génératrice) in \( G(k) \) in the sense of Bruhat and Tits (see (6.1.1) in their initial big IHES paper on what came to be called Bruhat–Tits theory). This is very useful in the calculations that underlie Bruhat–Tits theory.

**Remark 3.7.** It is natural to wonder if the reduced root system \( k\Phi' \) consisting of non-multipliable roots in the \( k \)-root system \( k\Phi \) coincides with the root system of a split connected reductive \( k \)-subgroup of \( G \) containing \( S \) as a maximal torus. This issue arose in the handout on Cartan’s connectedness theorem, where we settled the case of \( k \)-rank equal to 1. The affirmative answer in general is give by Theorem 7.2 in the Borel–Tits IHES 27 paper on reductive groups, and a much simpler proof is given by [CGP, Thm. C.2.30] (specialized to the reductive case, where some technicalities simplify and the rank-1 calculations were already carried out in the handout on Cartan’s connectedness theorem).

As an application of the direct spanning of the group \( U_\Psi \) for saturated \( \Psi \) as in Proposition 3.3, we can prove:

**Corollary 3.8.** Let \( P \) be a minimal parabolic \( k \)-subgroup of \( G \) containing \( S \), and \( \Phi^+ = \Phi(P, S) \). For \( w \in kW = W(k\Phi) \), define

\[
\Phi^+_w = \{ a \in k\Phi^+ \mid w^{-1}.a \in k\Phi^+ \}, \quad \Phi'_w = k\Phi^+ - \Phi^+_w.
\]

For a representative \( n_w \in N_G(S)(k) \) of \( w \in kW \), we have

\[
P(k)n_wP(k) = U_{\Phi^+_w}(k) \times Z_G(S)(k) \times n_wU_k\Phi^+(k).
\]

In view of the direct spanning property of the groups \( U_{\Phi'_w} \) and \( U_k\Phi^+ \), this is proved by the exact same computation as reviewed in the split case in §1. Of course, unlike in the split case, in the general case there is no combinatorial recipe for the dimensions of the affine spaces \( U_{\Phi'_w} \) or \( U_k\Phi^+ \) since the dimensions of the root groups \( U_a \) are unknown!

### 4. Simple factors

We proved in class (see Theorem 8.2.5) that if \( G \) is a connected semisimple \( k \)-group and \( \{G_i\} \) is its set of minimal nontrivial smooth connected normal \( k \)-subgroups then: \( \{G_i\} \) is finite, the \( G_i \)'s pairwise commute, each \( G_i \) is \( k \)-simple (i.e., has no nontrivial smooth connected proper normal \( k \)-subgroup), and the multiplication map

\[
\prod G_i \to G
\]

is a central isogeny. The essential case in the proof was the \( k \)-split case (such as over \( k_s \)), from which the general case was descent by Galois descent from \( G_{k_s} \).

In the split case we saw an additional property: if \( T \subset G \) is a split maximal \( k \)-torus and \( T_i = T \cap G_i \) (a split maximal \( k \)-torus of \( G_i \); see Lemma 4.1 below) then via the decomposition \( X(T)_{Q} = \prod X(T_i)_{Q} \), the root systems \( \Phi(G_i, T_i) \) are the irreducible components of \( \Phi(G, T) \). In particular, \( G \) is absolutely simple if and only if \( \Phi(G, T) \) is irreducible. We shall establish an analogue in general, but must be attentive to the fact that some \( G_i \) might be \( k \)-anisotropic (and so do not contribute to \( k\Phi \); they will lie in \( Z_G(S)! \)). As a preliminary step, we record two lemmas.
Lemma 4.1. Let $G$ be a connected reductive $k$-group and $N$ a smooth connected normal $k$-subgroup of $G$ (so $N$ is connected reductive). Let $S$ be a maximal $k$-split torus of $G$, and $T$ a maximal $k$-torus of $G$. Then $S_N := (S \cap N)_\text{red}$ is a maximal $k$-split torus of $N$, and $T \cap N$ is a maximal torus of $N$.

Proof. By the $G(k)$-conjugacy of all such $S$ and the normality of $N$ in $G$, the dimension of $S_N$ is independent of $S$. Thus, for the assertion concerning $S$ it suffices to find one $S$ for which $S_N$ is maximal as a $k$-split torus in $N$. Pick a maximal $k$-split torus $S'$ in $N$, so it lies in a maximal $k$-split torus $S''$ of $G$. Clearly $S''_N = S'$ by maximality of $S'$ in $N$, so we are done.

Next, we show that $T \cap N$ is a maximal $k$-torus of $N$. For this purpose we may extend scalars so that $k = \overline{k}$, and then we can apply the preceding case to see that $(T \cap N)_\text{red}$ is a maximal torus of $N$. It then remains to check that $T \cap N$ is smooth and connected. Since $T = Z_G(T)$, clearly $T \times (T \cap N)$ is the centralizer of $T$ in $T \times N$. But a torus centralizer in a smooth connected affine group is always smooth and connected, so we are done. ■

Lemma 4.2. For any smooth connected normal $k$-subgroup $M$ in a connected reductive $k$-group $H$ and for any maximal $k$-torus $T$ in $M$, there is an almost direct product decomposition $T = T' \cdot Z$ where $T' = T \cap \mathcal{D}(M)$ is a maximal $k$-torus in $\mathcal{D}(M)$ and $Z$ is the maximal subtorus of $T$ that centralizes $M$.

Proof. By Galois descent we may assume $k = k_s$, so by the split theory such $M$ is generating by some of the (commuting) simple factors of $H$ and a subtorus of the maximal central $k$-torus of $H$. Since $T$ compatibly decomposes as an almost direct product of the maximal central torus and maximal tori in each of the simple factors, the assertion is evident. ■

Here is the main result:

Proposition 4.3. Let $G$ be a connected reductive $k$-group, $S$ a maximal $k$-split torus, and $k\Phi = \Phi(G,S)$. Let $S'$ be the maximal $k$-split torus $(S \cap \mathcal{D}(G))_\text{red}$ of $\mathcal{D}(G)$. Assume $S' \neq 1$, or equivalently $S$ is non-central in $G$. For each irreducible component $\Psi$ of $k\Phi$, let $N_\Psi$ be the nontrivial smooth connected $k$-subgroup generated by $\{U_a\}_{a \in \Psi}$.

1. The $k$-group $G$ is generated by $Z_G(S)$ and the $N_\Psi$’s.
2. Each $N_\Psi$ is semisimple and normal in $G$.
3. The maximal $k$-split torus $S_\Psi := (S \cap N_\Psi)_\text{red}$ of $N_\Psi$ is nontrivial, $\prod_\Psi S_\Psi \to S'$ is an isogeny, and the $k$-root system of $N_\Psi$ with respect to $S_\Psi$ is naturally identified with $\Psi$ via the natural isomorphism $X(S')_\mathbb{Q} \simeq \prod_\Psi X(S_\Psi)_\mathbb{Q}$.
4. Each $k$-isotropic normal connected semisimple $k$-subgroup of $G$ contains some $N_\Psi$.
5. No $N_\Psi$ contains a nontrivial smooth connected proper normal $k$-subgroup.

Proof. Each $k$-subgroup $N_\Psi$ is clearly normalized by $Z_G(S)$, and Lie algebra considerations imply that $G$ is generated by $Z_G(S)$ and the $N_\Psi$’s. For a choice of $\Psi$ and roots $a \in \Psi$ and $b \in \Phi - \Psi$, $U_a$ commutes with $U_b$ by Proposition 3.5 since $ma + nb \notin k\Phi$ for any positive integers $m$ and $n$. Since $G$ is generated by $Z_G(S)$ and the root groups $\{U_b\}_{b \in \Phi}$, we see that $N_\Psi$ is a (smooth connected nontrivial) normal $k$-subgroup of $G$, so $N_\Psi$ is reductive.

Applying Lemma 4.2 to $G$ and its normal subgroup $N_\Psi$ and a maximal $k$-torus $T$ of $G$ containing $S$, the almost direct product decomposition of $T$ passes to one on maximal split
subtori (with the help of Lemma 4.1) we see that $S$ is an almost direct product of the maximal $k$-split torus $(S_{\Psi} \cap \mathcal{D}(N_{\Psi}))^{0}_{\text{red}}$ of $\mathcal{D}(N_{\Psi})$ and the maximal $k$-subtorus of $S$ that centralizes $N_{\Psi}$. Thus, $(S,U_a) = (S_{\Psi}, U_a) \subseteq \mathcal{D}(N_{\Psi})$ for all $a \in \Psi$, yet $(S,U_a) = U_a$ by the known $S$-equivariant structure of $U_a$ in the non-multipliable case and of $U_{2a}$ and $U_a/U_{2a}$ in the multipliable case (see Proposition 3.1), so $N_{\Psi}$ is perfect (thus semisimple) and $S_{\Psi} \neq 1$.

We have established (1), (2), and the first assertion in (3).

Recall that a smooth connected normal $k$-subgroup of a smooth connected normal $k$-subgroup of $G$ is normal in $G$ as follows from the general structure of smooth connected normal subgroups of connected reductive groups over $\overline{k}$, via irreducible components of root systems. Thus, a minimal nontrivial smooth connected normal $k$-subgroup $N$ of $G$ cannot contain a smooth connected proper $k$-subgroup that is normal in $N$.

Let $N$ be a non-commutative smooth connected normal $k$-subgroup of $G$, so the $k$-subgroup $\mathcal{D}(N)$ of $N$ is nontrivial and perfect and normal in $G$. Thus, if such an $N$ is minimal among the non-commutative smooth connected normal $k$-subgroups of $G$ then it is perfect and hence is contained in $\mathcal{D}(G)$ (which is itself perfect). But $S$ is an almost direct product of $S'$ and the maximal $k$-split central torus of $G$. Each $N_{\Psi}$ lies in $\mathcal{D}(G)$ by perfection, so the $k$-root system of $G$ with respect to $S$ is identified with that of $\mathcal{D}(G)$ with respect to $S'$. We may therefore replace $G$ with $\mathcal{D}(G)$ (and $S$ with $S' \neq 1$), so $G$ is semisimple and $S' = S$. In particular, by the split theory over $k_s$, every nontrivial normal smooth connected $k$-subgroup of $G$ is semisimple.

Let $\{N_i\}_{i \in I}$ be the set of minimal nontrivial smooth connected normal $k$-subgroups of $G = \mathcal{D}(G)$, so every $N_i$ is semisimple. By the split theory over $k_s$, such $N_i$ are almost direct products of $\text{Gal}(k_s/k)$-orbits among the minimal normal smooth connected $k_s$-subgroups of $G_{k_s}$. This description over $k_s$ implies the following properties if the $N_i$'s: they pairwise commute, the product homomorphism

$$
\pi : N = \prod_{i \in I} N_i \longrightarrow G
$$

is a central isogeny, and every normal smooth connected $k$-subgroup of $G$ is generated by $\{N_i\}_{i \in J}$ for a unique subset $J \subseteq I$. Note also that no $N_i$ contains a nontrivial smooth connected proper normal $k$-subgroup, due to the minimality of $N_i$ in $G$. We shall eventually see that the $k$-isotropic $N_i$'s are precisely the $N_{\Psi}$'s.

For $i \in I$, let $S_i = (S \cap N_i)^{0}_{\text{red}}$, a maximal $k$-split torus of $N_i$. Let $I^s$ be the set of $i \in I$ such that $N_i$ is $k$-anisotropic. Thus, $S_i = 1$ precisely for $i \in I^s$, and the pair $(N_i, S_i)$ has a non-empty $k$-root system for $i \in I - I^s$ (since otherwise the nontrivial split torus $S_i$ is central in the semisimple $N_i$, an absurdity). We claim that $S$ is an isogenous quotient of the maximal $k$-split torus $\prod_{i \in I - I^s} S_i$ of $N$. Choose a maximal $k$-torus $T$ of $G$ containing $S$, so by Lemma 4.1 we see that $T_i := T \cap N_i$ is a maximal $k$-torus of $N_i$ that contains $S_i$. The product $\prod T_i$ is a maximal $k$-torus of $N$, so its image in $G = \pi(N)$ is a maximal $k$-torus of $G$. This forces $\prod T_i \rightarrow T$ to be surjective, so the maximal split $k$-subtorus $\prod_{i \in I} S_i = \prod_{i \in I - I^s} S_i$ of $\prod T_i$ maps onto the maximal split $k$-subtorus $S$ of $T$.

The isogeny $\prod_{i \in I - I^s} S_i \rightarrow S$ implies that: $N_i \subseteq Z_G(S)$ for every $i \in I^s$, $\pi$ induces an injective homomorphism of $X_{*,(\prod_{i \in I - I^s} S_i)} = \prod_{i \in I - I^s} X_{*}(S_i)$ into $X_{*}(S)$ whose image is a
subgroup of finite index, and $I - I^2$ is non-empty. Thus, for any subset $J \subseteq I^*$, the $k$-subgroup $N_J := \langle N_j \rangle_{j \in J}$ is $k$-anisotropic (because we can apply the preceding considerations to $N_J$ in the role of $G$ to deduce that $N_J$ is $k$-anisotropic).

For any $\lambda \in \prod_{i \in I - I^2} X_*(S_i)$, the restriction of $\pi$ to $U_N(\lambda)$ is an isomorphism onto $U_G(\pi \circ \lambda)$ since the kernel of $\pi$ is central (forcing $\ker \pi \subseteq Z_N(\lambda)$). Hence, the set of root groups of $G$ relative to $S$ is the (disjoint) union of the set of root groups of $N_i$ relative to $S_i$ over all $i \in I - I^2$. Thus, $\Phi$ is the direct sum of the non-empty $k$-root systems $\Phi_i := \Phi(N_i, S_i)$ for $i \in I - I^2$ compatibly with the decomposition $X(S)_Q = \prod_{i \in I - I^2} X(S_i)_Q$.

We claim that for each $i \in I - I^2$, the $k$-root system $\Phi_i$ in $X(S_i)_Q \neq 0$ is irreducible. Granting this, $\{\Phi_i\}_{i \in I - I^2}$ is the set of irreducible components of the root system $k\Phi$ in the $\mathbb{Q}$-vector space $X(S)_Q = \prod_{i \in I - I^2} X(S_i)_Q$ and (by minimality) the nontrivial perfect normal $k$-subgroup $N_{\Phi_i}$ of $N_i$ must exhaust $N_i$ for each $i \in I - I^2$. This would establish (3) and (4), as well as (5). Thus, we can replace $(G, S)$ with the $k$-isotropic $(N_{i_0}, S_{i_0})$ for each $i_0 \in I - I^2$ separately. Now the $k$-isotropic connected semisimple $G$ has no nontrivial smooth connected normal $k$-subgroup apart from itself. In such cases it suffices to prove that the non-empty root system $k\Phi$ is irreducible.

Let $\Psi$ be an irreducible component of $k\Phi$; our aim is to prove that $\Psi = k\Phi$. Since $N_\Psi$ is a nontrivial smooth connected normal $k$-subgroup of $G$, $N_\Psi = G$. Suppose there exists $b \in k\Phi - \Psi$, so the nontrivial smooth connected unipotent $k$-group $U_b$ centralizes $N_\Psi = G$ and hence is central in $G$. This contradicts that $G$ is reductive, so $\Psi = k\Phi$. ■

The preceding proof yields immediately:

**Corollary 4.4.** Assume $G$ is semisimple. Let $\{\Phi_i\}$ be the set of irreducible components of $k\Phi$, and let $\{N'_j\}$ be the set of $k$-anisotropic minimal nontrivial normal connected semisimple $k$-subgroups of $G$. The multiplication homomorphism

$$\prod N'_j \times \prod N_{\Phi_i} \to G$$

is a central isogeny, and every normal connected semisimple $k$-subgroup $N$ of $G$ is an almost direct product among the $N'_j$’s and the $N_{\Phi_i}$’s.

The main point of this corollary is that the isotropic minimal nontrivial smooth connected normal $k$-subgroups of $\mathcal{D}(G)$ are constructed from the root groups for the roots in an irreducible component of $k\Phi$. 