Math 249B. Levi subgroups

1. Introduction

In the theory of Lie algebras in characteristic 0, there is an important result due to Levi and Malcev: if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field k of characteristic 0 and $\mathfrak{r}$ is its radical (i.e., maximal solvable Lie ideal), so $\mathfrak{g}/\mathfrak{r}$ is the maximal semisimple quotient of $\mathfrak{g}$, then there is a homomorphic section to $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{r}$. This says that there exists a semisimple Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \ltimes \mathfrak{r} = \mathfrak{g}$. The theorem is more precise, showing (among other things) that all possibilities for $\mathfrak{h}$ are carried into each other by automorphisms of $\mathfrak{g}$. Details are given in Bourbaki LIE I, §6.8. Such an $\mathfrak{h}$ is usually called a "Levi factor" of $\mathfrak{g}$.

In the context of linear algebraic groups, we have a related notion using the (geometric) unipotent radical rather than the radical (since at the level of algebraic groups there is better control of the role of commutative objects, unlike at the level of Lie algebras):

Definition 1.1. Let G be a linear algebraic group over a field k. A *Levi k-subgroup* of G is a closed linear algebraic k-subgroup $L \subset G$ such that $L_{\overline{k}} \to G_{\overline{k}}/\mathscr{R}_u(G_{\overline{k}})$ is an isomorphism (equivalently, $L_{\overline{k}} \ltimes \mathscr{R}_u(G_{\overline{k}}) = G_{\overline{k}}$).

Example 1.2. The dynamic constructions provide many examples, as follows. Let G be a connected reductive k-group and $\lambda : \mathbf{G}_m \to G$ a k-homomorphism, so

$$P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$$

as linear algebraic groups with torus centralizer $Z_G(\lambda)$ that must be connected reductive and $U_G(\lambda)$ is connected and unipotent. The formation of the dynamic constructions commutes with extension of the ground field (due to their definitions in terms of representing functors), so we see that $U_G(\lambda)_{\overline{k}}$ is the unipotent radical of $P_G(\lambda)_{\overline{k}}$. Hence, $Z_G(\lambda)$ is a Levi k-subgroup of $P_G(\lambda)$ and the subgroup $U_G(\lambda)$ is a k-descent of $\mathscr{R}_u(P_G(\lambda)_{\overline{k}})$.

The most concrete instance of this arises for $G = \mathrm{GL}_n$ and

$$\lambda : t \mapsto \mathrm{diag}(t^{e_1}, \dots, t^{e_n})$$

with $e_1 \geq \cdots \geq e_n$, in which case $P_G(\lambda)$ is the standard parabolic corresponding to the partition of $\{1, \dots, n\}$ based on distinct values among the e_i's and $Z_G(\lambda)$ is the usual corresponding subgroup $\prod \mathrm{GL}_{m_j}$ of "block matrices" (where the m_j's count the number of repetitions of a common value among the e_i's).

Later in this course we will show for general connected reductive G over general fields k that the k-subgroups $P_G(\lambda)$ are parabolic and moreover that *all* parabolic k-subgroups of G arise in this manner (for some k-homomorphism λ). Hence, it will follow that a parabolic k-subgroup P in a connected reductive k-group G *always* admits a Levi k-subgroup, and moreover that $\mathscr{R}_u(P_{\overline{k}})$ descends (obviously uniquely) to a unipotent smooth connected normal k-subgroup $U \subset P$. (In fact, we will show further that every Levi k-subgroup in P arises as $Z_G(\lambda)$ for some λ satisfying $P_G(\lambda) = P$, from which it will follow with more work that the set of Levi k-subgroups of P is a principal homogeneous space for the conjugation action by $U(k)$). This is remarkable in positive characteristic due to the counterexamples discussed below.

2. Existence results and counterexamples

The results discussed below will never be used in this course, but we discuss them for purposes of cultural awareness. In characteristic 0 there is a group analogue of the Levi–Malcev theorem:

Theorem 2.1 (Mostow)**.** *Let H be a linear algebraic group over a field k of characteristic* 0*, and let $U \subset H$ be the unique linear algebraic k-subgroup descending $\mathscr{R}_u(H_{\overline{k}})$* (*as exists by Galois descent since k is perfect*)*. Then H admits a Levi k-subgroup L and the set of such L is a homogeneous space for the conjugation action of $U(k)$.*

The proof of this theorem lies beyond our present stage of development, as it rests on the structure theory of connected reductive groups and the complete reducibility of their representation theory in characteristic 0. A modern proof (using Hochschild cohomology) is given in Proposition 5.4.1 of my paper "Reductive group schemes" in the first volume of the Proceedings of the 2011 Luminy summer school on SGA3.

The characteristic-0 hypothesis in Mostow's theorem is essential: over every algebraically closed field k of positive characteristic there are many counterexamples to the existence of Levi subgroups. The most concrete class of such examples is given by the linear algebraic k-group "corresponding" to the group of k-points

$$\mathrm{SL}_n(W_2(k)),$$

where $n \geq 2$ and $W_2(k) = W(k)/(p^2)$ is the artin local ring of length-2 Witt vectors. (Here, SL_n can be replaced with any Chevalley group.) The rigorous definition of this example as a k-group via a functor of points is given in A.6 of the book "Pseudo-reductive Groups", where it is proved that such groups never admit Levi subgroups.

The key idea underlying the proof of absence of Levi subgroups for such examples is to analyze tori and "root groups" (going *beyond* the reductive setting to incorporate groups such as $\mathrm{SL}_n(W_2(k))$) to exploit that the reduction map $q : W_2 \twoheadrightarrow \mathbf{G}_a$ has no homomorphic section. (Any such section $\mathbf{G}_a \to W_2$ to q must land inside the underlying reduced scheme $W_2[p]_{\mathrm{red}}$ of the p-torsion of W_2, but this reduced scheme is $\ker q$, and a *section* to q cannot land in there.)

To explain how the study of q rules out the existence of a Levi subgroup, we will abuse notation and speak in terms of k-points in order to simplify notation; to be rigorous, one should work with functors (as is done in A.6 of "Pseuso-reductive Groups"). The reduction map $W_2(k) \twoheadrightarrow k$ has nontrivial square-zero kernel $pW_2(k) = k$ on which $W_2(k)$-multiplication acts through composing Frobenius on $W_2(k)$ with reduction modulo p. This induces a quotient map of linear algebraic groups

$$\mathrm{SL}_n(W_2(k)) \twoheadrightarrow \mathrm{SL}_n(k)$$

whose (scheme-theoretic) kernel is the vector group $\mathrm{Mat}_n(k)^{\mathrm{Tr}=0}$ of traceless $n \times n$ matrices. (Since this kernel is commutative, the quotient term $\mathrm{SL}_n(k)$ naturally acts upon the kernel as an additive k-group. This is *not* the usual conjugation action of $\mathrm{SL}_n(k)$ on traceless $n \times n$ matrices, but rather is the composition of the Frobenius on $\mathrm{SL}_n(k)$ with such conjugation.)

Suppose there is a k-homomorphic section $\sigma : \mathrm{SL}_n(k) \to \mathrm{SL}_n(W_2(k))$. With some work, one can show that σ can be arranged to carry the diagonal torus $(k^\times)^n$ into the diagonal torus $(k^\times)^n \subset (W_2(k)^\times)^n$ (latter inclusion defined by the Teichmüller decomposition $W_2(k)^\times = k^\times \times (1+pW_2(k))$). This identification of copies of $(k^\times)^n$ must be via the *identity map* because σ is a section, so one can thereby deduce via dynamic methods that σ must be compatible with "root groups" for the nontrivial weights for these maximal tori acting on the corresponding Lie algebras. But such "root groups" in the non-reductive $\mathrm{SL}_n(W_2(k))$ (defined very generally by *dynamic methods* and uniquely characterized *without* reference to dynamics) are copies of the 2-dimensional k-group W_2, so (between root groups for a common root) σ restricts to a k-homomorphic section of the reduction map $q : W_2 \twoheadrightarrow \mathbf{G}_a$. We saw that there is no such section, so no such σ exists.

Remark 2.2. If H is a linear algebraic group over a perfect field then by Galois descent $\mathscr{R}_u(H_{\overline{k}})$ descends to a (unipotent smooth connected normal) k-subgroup $U \subset H$. Galois descent is not applicable if k is not perfect (e.g., local or global function field). Nonetheless, we noted in Example 1.2 (and will prove later in the course via dynamic methods) that for *any* field k parabolic k-subgroups P of connected reductive k-groups always admit a k-descent of their geometric unipotent radical.

To appreciate how remarkable it is that $\mathscr{R}_u(P_{\overline{k}})$ descends to a k-subgroup of P without any perfectness restrictions on k (affirming the miraculous nature of the theory of connected reductive groups), we note that for *every* imperfect field k there are *many* examples of connected linear algebraic k-groups G for which $\mathscr{R}_u(G_{\overline{k}})$ fails to descend to a k-subgroup of G. The most concrete class of such examples are Weil restrictions $G = \mathrm{R}_{k'/k}(G')$ for nontrivial purely inseparable finite extensions k'/k and nontrivial connected reductive k'-groups G' (see Example 1.6.1 in "Pseudo-reductive Groups" for the justification). The purpose of the theory of pseudo-reductive groups is exactly to handle this phenomenon.