Math 249B. Homework 7

1. Read the handout on L-functions and Dirichlet density.

2. Recall that the idelic norm $\|\cdot\|_K : \mathbf{A}_K^\times/K^\times \to \mathbf{R}_{>0}^\times$ has rather different structure in the number field and function field cases: it is surjective for number fields whereas its image in the function field case is infinite cyclic (and in fact is $q^{\mathbf{Z}}$ where q is the size of the constant field of K). In both cases the kernel is compact, by the synthesis of the main finiteness theorems for unit groups and class groups.

Using the nature of the idelic norm in the two cases, prove that all open subgroups of the idele class group $\mathbf{A}_K^\times/K^\times$ have finite index in the number field case (hint: use an archimedean place to define a continuous section to the idelic norm map) but show by example that this is always false in the function field case. Formulate a salvage which characterizes (in terms of the idelic norm map) which open subgroups have finite index in the function field case.

3. Let L/K be a finite unramified extension of (non-archimedean) local fields. Prove that the norm map $\mathscr{O}_L^\times \to \mathscr{O}_K^\times$ is surjective by using successive approximation (starting at the residue field level, and building up using that $(1+\mathfrak{m}^i)/(1+\mathfrak{m}^{i+1}) \simeq \mathfrak{m}^i/\mathfrak{m}^{i+1}$ for all $i > 0$). You have to use unramifiedness in a crucial way.

4. A *Hecke character* (or *grossencharacter* if you like German) for a global field K is a continuous homomorphism $\chi : \mathbf{A}_K^\times \to \mathbf{C}^\times$ such that $\chi(K^\times) = 1$. (Equivalently, it is a continuous homomorphism $\mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$.) The original source of interest in these is due to class field theory, as well as certain examples arising from the theory of complex multiplication for elliptic curves.

(*i*) Using polar form, prove that there is a neighborhood of 1 in $\mathbf{C}^\times$ that contains no nontrivial subgroups. As an application, show that a continuous homomorphism $\Gamma \to \mathbf{C}^\times$ for a profinite group Γ must kill an open (hence finite-index) subgroup. Show this is false if we relax "profinite" to "compact". If you know Lie theory, find a more sophisticated argument to prove the same with $\mathbf{C}^\times$ replaced with an arbitrary Lie group.

(*ii*) Let χ be a Hecke character for K. Prove that for all but finitely many non-archimedean places v of K, $\chi|_{\mathscr{O}_v^\times} = 1$. (This does not require that $\chi(K^\times) = 1$.) For reasons related to class field theory, one says that χ is "unramified at v" when this local triviality condition holds. Using compactness of the norm-1 idele class group, prove that $\chi = \psi\|\cdot\|_K^s$ for some $s \in \mathbf{C}$ and some Hecke character ψ that is valued in the unit circle. In other words, the most important Hecke characters are the ones valued in the unit circle. (This includes the Hecke characters of finite order!) In the number field case show that s is uniquely determined (hint: look at $|\chi|$), but that in the function field case it is only unique modulo $(2\pi i/\log q)\mathbf{Z}$ and we can arrange for ψ to have finite order by changing s if necessary.

(*iii*) Using class field theory, explain why composing with the Artin map $\psi_K : \mathbf{A}_K^\times/K^\times \to G_K^{\mathrm{ab}}$ defines a bijection between the set of continuous homomorphisms $G_K \to \mathbf{C}^\times$ and the set of finite-order Hecke characters of K. That is the Langlands program for GL_1.

(*iv*) For $K = \mathbf{Q}(i)$ show that $\mathbf{A}_K^\times/K^\times \simeq (K_\infty^\times \times \prod_{v\nmid\infty} \mathscr{O}_v^\times)/\mu$ as topological groups, where $\mu = \{\pm 1, \pm i\}$ is the finite group of roots of unity in K. Upon fixing an isomorphism $j : K_\infty \simeq \mathbf{C}$ as extensions of $\mathbf{R}$, show that $\alpha \mapsto j(\alpha_\infty/|\alpha_\infty|)^4$ on $K_\infty^\times \times \prod \mathscr{O}_v^\times$ defines an S^1-valued infinite-order Hecke character of K. Make an analogous construction for $\mathbf{Q}$, $\mathbf{Q}(\sqrt{-3})$, and any imaginary quadratic field with class number 1. It is not quite so simple to make such examples for other number fields (units and class groups create difficulties), but they do exist.