

MATH 249B. HOMEWORK 6

1. (i) Read §6–§8 in Chapter IV in Cassels-Fröhlich (and the rest if you are feeling inspired and have time; it is not necessary for following the rest of this course, but of course is crucial at such time as you intend to read up on certain proofs to be omitted later).

(ii) Read §2.1–§2.4 and §2.6–§2.7 in Chapter V of Cassels-Fröhlich to see how one makes a good theory of cohomology for profinite groups and their actions on discrete modules. To check your appreciation of the role of the topology in all of this, make sure you understand why for a profinite group G and the discrete G -module \mathbf{Q} we have $H^1(G, \mathbf{Q}) = 0$ in the sense of profinite group cohomology whereas if we just form cohomology for G as an abstract group then the corresponding degree-1 cohomology can be *nonzero* (e.g., with $G = \widehat{\mathbf{Z}}$).

(iii) If you are familiar with the theory of derived functors, prove that the category of discrete modules for a profinite group G has enough injectives (hint: For a general G -module N define N_{disc} to be its maximal “discrete” subgroup, which is to say the subgroup of elements on which G acts with open stabilizers. Prove a universal mapping property for this operation, and use it to prove that if I is an injective discrete $\mathbf{Z}[G]$ -module then I_{disc} is an injective object in the category of discrete G -modules.) Then use Grothendieck’s erasability criterion to prove that the right-derived functor of $M \rightsquigarrow M^G$ on the category of discrete G -modules is uniquely δ -functorially isomorphic to the “explicitly constructed” profinite G -cohomology theory built in §2 of Chapter V of Cassels-Fröhlich.

One virtue of the concrete construction is that it shows $H^p(G, M)$ is a torsion group for all $p > 0$ and all discrete G -modules M . Can you see this property using the derived functor approach?

2. Read the handout on interpretation of low-degree group cohomology and check whatever omitted details from it you feel like checking.

3. This exercise uses the additive and multiplicative Hilbert 90 in profinite group cohomology (as discussed in §2.6–§2.7 of Chapter V of Cassels-Fröhlich). Let K be a field, K_s/K a choice of separable closure, and $G = \text{Gal}(K_s/K)$ as a profinite group.

(i) Pick an integer n not divisible by $\text{char}(K)$ and assume that K contains a primitive n th root of unity. Let $\mu_n \subseteq K^\times$ be the subgroup of n th roots of unity in K . Prove that the diagram of discrete G -modules

$$1 \rightarrow \mu_n \rightarrow K_s^\times \xrightarrow{x^n} K_s^\times \rightarrow 1$$

is a short exact sequence, and that the connecting map in the long exact cohomology sequence induces an isomorphism

$$K^\times / (K^\times)^n \simeq \text{Hom}_{\text{cont}}(G, \mu_n).$$

Unravelling the construction of the connecting map, show that this isomorphism carries the class of a mod $(K^\times)^n$ to the homomorphism $G \rightarrow \mu_n$ defined by $g \mapsto g(a^{1/n})/a^{1/n}$ for any fixed n th root $a^{1/n}$ of a in K_s .

(ii) Use the isomorphism $K^\times / (K^\times)^n \simeq \text{Hom}_{\text{cont}}(G, \mu_n)$ from (i) along with infinite Galois theory to define a bijection between the set of subgroups $B \subseteq K^\times / (K^\times)^n$ and the set of abelian extensions K'/K with exponent n (but possibly infinite degree). Then show that this is exactly the correspondence $B \mapsto K(B^{1/n})$ as in Kummer theory (and so this gives a very quick proof of Kummer theory!).

(iii) Now assume $\text{char}(K) = p > 0$. Prove that the diagram of discrete G -modules

$$0 \rightarrow \mathbf{F}_p \rightarrow K_s \xrightarrow{x^p - x} K_s \rightarrow 0$$

is a short exact sequence, and carry out the Artin-Schreier analogues of what you did in (i) and (ii).