Math 249B. Homework 10

1. Let L/K be a finite abelian extension of global fields.

(*i*) By using the triviality of ψ_K on $K^\times$ and using local class field theory in a manner similar to our proof in class that $L \subseteq K_{\mathfrak{m}} \Leftrightarrow \mathfrak{f}_{L/K}|\mathfrak{m}$, show that if v_0 is a place of K (possibly archimedean) and $\alpha \in K^\times$ is a local norm for all $v \neq v_0$ (i.e., for all $v \neq v_0$ we have $\alpha \in \mathrm{N}_{K_v}^{L_w}(L_w^\times)$ for one, and hence any, $w|v$ in L) then α is also a local norm at v_0.

(*ii*) Assume $\mathrm{char}(K) \neq 2$. For nonsquare $a, b \in K^\times$ show that a is a norm from $L = K(\sqrt{b})$ if and only if $ax^2 + by^2 = z^2$ has a nonzero solution in K. By doing a similar argument over local fields, deduce with the help of (*i*) that if $Q(x, y, z)$ is a nondegenerate ternary quadratic form over K and $Q = 0$ has a nontrivial solution in K_v for all v away from some place v_0 then there is also a nontrivial solution in K_{v_0}. (In algebro-geometric language, this says that if C is a smooth projective and geometrically connected curve of genus 0 over K and if $C(K_v) \neq \emptyset$ for all $v \neq v_0$ then $C(K_{v_0}) \neq \emptyset$ as well.) The Hasse norm theorem in Exercise 2 below (applied to quadratic extensions of K) shows that in such cases there is a nontrivial solution in K as well; this is a step in the proof of the Hasse-Minkowski theorem for nondegenerate quadratic forms over any global field.

2. Let K be a global field. Let L/K be a finite Galois extension. Let $G = \mathrm{Gal}(L/K)$.

(*i*) Using Hilbert's Theorem 90 over the completions and residue fields of K, prove that $\mathrm{H}^1(G, \mathbf{A}_L) = 0$ and $\mathrm{H}^1(G, \mathbf{A}_L^\times) = 1$. (Hint: prove the analogue for each $\mathbf{A}_{L,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathscr{O}_{K_v}$ separately, and then pass to the direct limit.) Deduce that

$$\mathrm{H}^1(G, \mathbf{A}_L^\times / L^\times) \simeq \ker(\mathrm{H}^2(G, L^\times) \to \mathrm{H}^2(G, \mathbf{A}_L^\times)).$$

(*ii*) For each non-archimedeann place v of K that is unramified in L and each associated decomposition group $D(w|v) \simeq \mathrm{Gal}(L_w/K_v) \subseteq G$, prove by periodicity of cohomology for cyclic groups that $\mathrm{H}^2(D(w|v), \mathscr{O}_{L_w}^\times) = 1$. Upon choosing some w over each place v of K, deduce that

$$\mathrm{H}^2(G, \mathbf{A}_L^\times) \simeq \oplus_v \mathrm{H}^2(D(w|v), L_w^\times)$$

(direct sum, not direct product!) and hence that the natural restriction map $\mathrm{Br}(L/K) \to \prod_v \mathrm{Br}(L_w/K_v)$ lands inside of the *direct sum* $\oplus_v \mathrm{Br}(L_w/K_v)$, yielding an exact sequence

$$0 \to \mathrm{H}^1(G, \mathbf{A}_L^\times / L^\times) \to \mathrm{Br}(L/K) \to \oplus_v \mathrm{Br}(L_w/K_v)$$

with natural localization maps.

(*iii*) In the proofs of class field theory, one shows that $\mathrm{H}^1(G, \mathbf{A}_L^\times / L^\times) = 1$ whenever L/K is a cyclic extension (and eventually when it is any finite Galois extension at all), so in particular the localization map $\mathrm{Br}(L/K) \to \oplus_v \mathrm{Br}(L_w/K_v)$ is *injective* whenever L/K is cyclic (and eventually when it is arbitrary, so passing to the limit gives injectivity of $\mathrm{Br}(K) \to \oplus_v \mathrm{Br}(K_v)$). Let's exploit this in the cyclic case via the double-periodicity of cohomology for cyclic groups.

Recall that when G is any cyclic group of order n with a generator s we have a canonical isomorphism $\mathrm{H}^2(G, \mathbf{Z}) \simeq \mathrm{H}^1(G, \mathbf{Q}/\mathbf{Z}) \simeq (1/n)\mathbf{Z}/\mathbf{Z}$, with $1/n \bmod \mathbf{Z}$ going over to a generator $\theta_s \in \mathrm{H}^2(G, \mathbf{Z})$ whose cup product action defines the double-periodicity of Tate cohomology $\widehat{\mathrm{H}}^\bullet(G, \cdot)$. If $D \subseteq G$ is a subgroup of index m then use the definition of θ_s to check that the restriction map $\mathrm{H}^2(G, \mathbf{Z}) \to \mathrm{H}^2(D, \mathbf{Z})$ carries θ_s to θ_{s^m}.

Using this compatibility and the Tate cohomology isomorphism $\widehat{\mathrm{H}}^0 \simeq \widehat{\mathrm{H}}^2$ for cyclic groups to deduce that for cyclic L/K the natural map $K^\times / \mathrm{N}(L^\times) \to \oplus_v K_v^\times / \mathrm{N}(L_w^\times)$ is *injective* by reducing it to the analogous (proved) fact for Brauer groups! This is the *Hasse norm theorem* for cyclic extensions: if $\alpha \in K^\times$ is everywhere a local norm relative to a cyclic extension L/K then it is a norm from $L^\times$.

3. This exercise explores some aspects of norms in abelian externsions.

(*i*) Cyclicity is an essential hypothesis in the Hasse norm theorem. Consider $K = \mathbf{Q}$ and $L = \mathbf{Q}(\sqrt{13}, \sqrt{17})$. Show that each induced local extension is quadratic, unramified away from 13 and 17, and that -1 is a local square in L at 13 and 17. Using surjectivity of the local norm on units in the unramified case, deduce that -1 is a local norm from L to $\mathbf{Q}$. But prove that $-1 \notin \mathrm{N}_{\mathbf{Q}}^L(L^\times)$! (Hint: By using the Galois-theoretic description of the norm, show that a global norm for $L/\mathbf{Q}$ must be everywhere a local square, and hence be a square in $\mathbf{Q}$! Interestingly, by using more sophisticated considerations one can show that there are even rational squares which are not global norms from L, such as 25 and 49, yet all rational squares are everywhere local norms from L.)

Remark. In more geometric terms, upon writing out $\mathrm{N}_{\mathbf{Q}}^{L} : L \to \mathbf{Q}$ in suitable $\mathbf{Q}$-linear coordinates, this says that the smooth projective hypersurface

$$((x^2+13y^2)-17(z^2+13w^2))^2-4\cdot 13(xy-17zw)^2+u^4=0$$

of degree 4 in $\mathbf{P}^4_{\mathbf{Q}}$ has no $\mathbf{Q}$-points yet it has a rational point over each completion of $\mathbf{Q}$. In other words, this is an example of the failure of the so-called local-to-global principle (or "Hasse principle"), which does hold for smooth projective hypersurfaces of degree 2 (by the Hasse-Minkowski theorem). There are even examples of failure of the local-to-global principle in degree 3 in few variables, the most famous being Selmer's example of $3x^3+4y^3+5z^3=0$, but it is known that for cubic forms in 10 or more variables the local-to-global principle does hold and this is optimal in the sense that there are counterexamples in 9 variables. Needless to say, *justifying* a counterexample to the local-to-global principle requires some real work, since congruential obstructions cannot be found!

(ii) Prove that there does not exist a cyclic extension $L/\mathbf{Q}$ of degree 8 that is unramified at 2. (Hint: if such an extension exists, show that 16 is a local norm at all places away from 2, using that 16 is an 8th power in any field in which one of 2, -2, or -1 is a square. Then use Exercise 1(i) to deduce that 16 is a local norm at 2, and get a contradiction via unramifiedness at 2.) It turns out that one can make a cyclic extension $L/\mathbf{Q}$ of degree 16 which induces the unramified extension of $\mathbf{Q}_2$ with degree 8.

Note that by the Kronecker-Weber theorem, these facts about cyclic extensions of $\mathbf{Q}$ (or degrees 8 or 16 with no ramification at 2) must amount to a concrete statement about $(\mathbf{Z}/N\mathbf{Z})^{\times}$'s. What is this fact?

Remark. The above shows that an unramified continuous character $\chi_2 : G_{\mathbf{Q}_2} \to \mathbf{C}^{\times}$ with order 8 *cannot* extend to a global continuous character $\chi : G_{\mathbf{Q}} \to \mathbf{C}^{\times}$ with order 8, but it does extend to one with order 16. In Chapter X of Artin–Tate they solve the delicate but important general problem of determining when a given finite set of continuous characters $\chi_v : G_{K_v} \to \mathbf{C}^{\times}$, say with respective orders n_v, for v in a finite set S of places of a global field K extend to a common global continuous character $\chi : G_K \to \mathbf{C}^{\times}$ (always affirmative), and especially with order *as small as possible*, namely $\mathrm{lcm}(n_v)_{v\in S}$ (this is usually possible, but sometimes one misses this by a factor of 2). By class field theory this amounts to a close topological study of the continuous injective homomorphism $\prod_{v\in S} K_v^{\times} \to \mathbf{A}_K^{\times}/K^{\times}$ (which is *not* an embedding when $|S|>1$), and it turns out that there are difficulties precisely when K is a number field and S contains a 2-adic place. This is closely tied up with the problem of whether $\alpha \in K^{\times}$ being everywhere locally an nth power forces it to lie in $(K^{\times})^n$ (HW1, Exercise 1!), whose solution is also fully explained in Chapter X of Artin–Tate.