Math 249B. Galois groups and abelianizations

As a general reference on field theory, we recommend Zariski-Samuel's *Commutative Algebra*, vol 1. As a general reference on topological groups, we recommend the exercises on topological groups in Munkres' *Topology* (be careful that in general formation of quotient topologies doesn't commute with formation of product topologies, so some care is needed). As a general reference on profinite groups, we recommend §1 in Chapter V in Cassels-Fröhlich (and the references therein).

1. Nonsense

Let $\varphi : K \to E$ be an extension of fields, equipped with separable closures K_s/K and E_s/E.

Lemma 1.1. *There exists a field map $\iota : K_s \to E_s$ over φ, and the resulting map of groups $\mathrm{Gal}(E_s/E) \to \mathrm{Gal}(K_s/K)$ is continuous. All such ι's are related via composition with $\mathrm{Aut}(K_s/K)$. Thus, the natural map $\mathrm{Aut}(E_s/E)^{\mathrm{ab}} \to \mathrm{Aut}(K_s/K)^{\mathrm{ab}}$ depends only on the given embedding φ and not on the auxiliary choice of ι.*

Proof. Let $L \subseteq E_s$ denote the set of all elements which are separable algebraic over $\varphi(K)$. Since E_s is a separably closed field, any separable polynomial over $\varphi(K)[X] \subseteq E_s[X]$ factors into linears over E_s. Thus, L is a separable algebraic closure of $\varphi(K)$. Since $\varphi : K \to E$ induces an isomorphism of K onto $\varphi(K)$, it follows by general business with Zorn's Lemma that one has "uniqueness" of separable (or better: algebraic) closures, which is to say that there exists an isomorphism ι between K_s and L over $\varphi : K \simeq \varphi(K)$. If ι_1, ι_2 are two such embeddings, then they both induce K-isomorphisms from K_s to L, whence $\sigma = \iota_1^{-1} \circ \iota_2$ makes sense as a K-automorphism of K_s. Obviously $\iota_2 = \iota_1 \circ \sigma$.

It remains to show that for any such choice of ι, the naturally induced map of groups

$$\mathrm{Gal}(E_s/E) \to \mathrm{Gal}(K_s/K)$$

is *continuous* for their natural topologies. As with any group homomorphism between topological groups, it suffices to check continuity at the identity element (why?). Recalling the base of open subgroups around the identity in $\mathrm{Gal}(K_s/K)$, we just have to show that if $x_1, \dots, x_n \in K_s$ are elements then there exist elements $y_1, \dots, y_m \in E_s$ such that if $\sigma \in \mathrm{Gal}(E_s/E)$ fixes $y_1, \dots, y_m$ then the restriction of σ to K_s (via ι) fixes $x_1, \dots, x_n$. Just take $y_j = \iota(x_j)$! ■

We now consider a general situation that will arise later in the theory of Weil groups. Let Γ be a profinite group and suppose we are given a short exact sequence of topological groups

$$1 \to I \to \Gamma \to \widehat{\mathbf{Z}} \to 1. \tag{1.1}$$

Recall that *exactness* here means not only exactness on underlying groups but also that $\widehat{\mathbf{Z}}$ inherits the quotient topology from Γ and I inherits the subspace topology from Γ. Since $\widehat{\mathbf{Z}}$ is Hausdorff, it follows that the kernel I must be a *closed* subgroup of Γ. In particular, I is compact Hausdorff. Since profinite groups (such as Γ and $\widehat{\mathbf{Z}}$) are *always* compact Hausdorff, in order to never have to worry too much about the topological business it is convenient to record:

Lemma 1.2. *Let $f : G \to H$ be a continuous map between compact Hausdorff topological groups. If f is surjective, then H inherits the quotient topology from G. If f is injective, then G inherits the subspace topology from H. In particular, if f is bijective then f is a homeomorphism.*

Moreover, any closed subgroup of a profinite group is a profinite topological group with respect to the induced subspace topology and any quotient of a profinite group by a closed normal subgroup is a profinite topological group with respect to the induced quotient topology.

Proof. The second part is a consequence of the general nonsense about profinite groups; see the chapter on profinite groups in Cassels-Fröhlich for discussion and references. As for the first part, this is a special case of the more general fact that a continuous surjection (resp. continuous injection) between compact Hausdorff spaces is a quotient map (resp. a homeomorphism onto the image). Since the closed subsets of a compact Hausdorff space are exactly the compact subsets, this is trivial.

■

Let $\pi : \Gamma \to \widehat{\mathbf{Z}}$ denote the projection, which uniquely factors through a *continuous* $\pi^{\mathrm{ab}} : \Gamma^{\mathrm{ab}} \to \widehat{\mathbf{Z}}$ (recall the definition of quotient topologies!), giving a short exact sequence

$$1 \to K \to \Gamma^{\mathrm{ab}} \to \widehat{\mathbf{Z}} \to 1 \tag{1.2}$$

where the closed subgroup K is necessarily profinite. This kind of structure is the model for local class field theory for non-archimedean fields and global class field theory for function fields, as we now indicate by means of examples.

2. Examples of Weil group data

Two examples to keep in mind are when $\Gamma = G_K = \mathrm{Gal}(K_s/K)$ for K either a local non-archimedean field or a global function field. Let k denote the finite residue field in the first case and the finite field of constants in the second case, with size q. In the local field case, the valuation on K uniquely extends to K_s, and the residue field $\overline{k}$ of the valuation ring on K_s is a separable closure of k (even an algebraic closure, since finite fields are perfect). In the global function field case, let $\overline{k}$ denote the separable (= algebraic) closure of k in the separably closed extension field K_s containing K (and hence containing k). Notice how this global function field setup is a special case of the situation in the first lemma above.

In both cases there is a natural map of compact groups

$$\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbf{Z}}, \tag{2.1}$$

and we claim this is *continuous and surjective*, whence it is a quotient map with profinite kernel, by the above lemma. In order to see the continuity and surjectivity of (2.1), let K' denote the maximal unramified extension of K inside of K_s in the local field case and let $K' = K \otimes_k \overline{k}$ in the global function field case. In the local field case, K' is a compositum of Galois extensions of K (by the theory of unramified extensions), so it is Galois over K. In the function field case, K' is a field by the lemma below and is naturally a subfield of K_s Galois over K (being naturally identified with the compositum of K and $\overline{k}$ inside of K_s via the natural K-map $K' \to K_s$ induced by $\overline{k} \hookrightarrow K_s$ over k). Thus, in both cases we have a natural surjection $\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(K'/K)$ and this is continuous by the first lemma above. In both cases there is a natural map

$$\mathrm{Gal}(K'/K) \to \mathrm{Gal}(\overline{k}/k) \tag{2.2}$$

(due to automatic continuity of automorphisms of finite extensions of local fields in the local field case), and its composite with continuous surjection $\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(K'/K)$ is clearly the map (2.1). Hence, we just have to check that (2.2) is continuous and surjective. We will even prove that this is a homeomorphism (it suffices to verify continuity and bijectivity, as source and target are profinite).

In the global function field case, continuity of (2.2) follows from the first lemma above and injectivity is obvious. Meanwhile, surjectivity in that case follows from the fact that any k-automorphism σ of $\overline{k}$ trivially extends to a K-automorphism of $K' = K \otimes_k \overline{k}$ via $1 \otimes \sigma$. In the local field case, we argue as follows. Note that we do *not* require the finiteness of the residue field. The theory of finite unramified extensions (e.g., as in the first chapter of Cassels-Fröhlich) shows that the "residue field" functor sets up an equivalence of categories between finite unramified extensions of K and finite separable extensions of the residue field k. In particular, if L/K is a finite unramified Galois extension then the residue field extension k_L/k is finite Galois with $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(k_L/k)$, and if $L, L' \subseteq K_s$ are two such extensions then $L \subseteq L'$ inside of K_s if and only if $k_L \subseteq k_{L'}$ inside of the residue field $\overline{k}$ of K_s. From this it follows that (2.2) is a continuous bijection.

Here is the lemma which was alluded to above:

Lemma 2.1. *Let K/k be an extension of fields and assume that k is algebraically closed in K; that is, the only elements of K which are algebraic over k are the elements of k. If k'/k is an algebraic extension, then $K \otimes_k k'$ is a field in which k' is algebraically closed. Meanwhile, if k is merely separably closed in K then for any separable algebraic k'/k the ring $K \otimes_k k'$ is a field in which k' is separably closed.*

Proof. Since every element in the tensor product is a finite sum of elementary tensors, we easily reduce to the case in which $[k' : k]$ is finite. By expressing the extension k'/k as a finite tower of primitive extensions, we easily reduce to the case in which $k' = k(\alpha)$. Let $f \in k[X]$ be the minimal polynomial of α over k, so $k' \simeq k[X]/f$ over k. When k is algebraically closed in K then f is irreducible in $K[X]$ since any factorization of f into monics in $K[X]$ must have coefficients algebraic over k and hence lying in k. Likewise, if k is separably closed in K and f is separable then f is irreducible in $K[X]$. Thus, in both cases $K \otimes_k k' \simeq K[X]/f$ is a field, separable over K when k'/k is separable (i.e., when f is separable).

It remains to show that k' is algebraically (resp. separably) closed in $K' \stackrel{\mathrm{def}}{=} K \otimes_k k'$ when k is algebraically (resp. separably) closed in K. Suppose some $x \in K'$ is algebraic (resp. separable) over k' but does not lie in k'. Let $k'' = k'(x) \subseteq K'$, a finite algebraic (resp. separable) extension of k'. Since $K' \otimes_{k'} k'' = K \otimes_k k''$, this is a field by what we have already shown (as k''/k is algebraic (resp. separable) and k is algebraically (resp. separably) closed in K). In particular, any subring of $K' \otimes_{k'} k''$ must be a domain. We will now contradict this fact. Since k'' is a k'-subalgebra of K' and k' is a field, the natural map

$$k'' \otimes_{k'} k'' \to K' \otimes_{k'} k''$$

is *injective*. Now I claim that $k'' \otimes_{k'} k''$ is *not* a domain! Indeed, $k'' = k'(x) \simeq k'[X]/g$ where $g \in k'[X]$ is the minimal polynomial of x, so g is irreducible with degree > 1 (here's where we use that $x \notin k'$ inside of K'). Thus,

$$k'' \otimes_{k'} k'' \simeq k'' \otimes_{k'} k'[X]/g \simeq k''[X]/g.$$

But $g \in k''[X]$ has degree > 1 and admits a root (namely $x \in k''$!), so it is not irreducible. Thus, the quotient ring $k''[X]/g$ cannot be a domain. ∎

3. More nonsense

We now return to the general situation in (1.1), with $\pi : \Gamma \to \widehat{\mathbf{Z}}$ the projection. Let $W = \pi^{-1}(\mathbf{Z})$. We call this the *Weil group* of the situation (1.1). In the case of global function fields, W consists of elements inducing an integral power of Frobenius on the separable closure $\overline{k}$ of the field of constants, and in the local non-archimedean case W consists of elements inducing an integral power of Frobenius on the residue field $\overline{k}$ of the valuation ring of K_s. Of course, W is a normal subgroup of Γ and it fits into an exact sequence of groups

$$1 \to I \to W \to \mathbf{Z} \to 0.$$

Recall that I is a compact (even profinite) topological group. We endow W with a topology by giving I its usual topology and declaring I to be an *open* subgroup of W. Let us see that this is a meaningful definition:

Lemma 3.1. *There is a unique structure of topological group on W with respect to which the normal subgroup I inherits its given topology from Γ and is open in W (so the quotient $W/I \simeq \mathbf{Z}$ inherits the discrete topology). If $\varphi \in W$ is an element mapping to $1 \in \mathbf{Z}$, then the natural map of sets $I \times \mathbf{Z} \to W$ determined by $(i,n) \mapsto i\varphi^n$ is a homeomorphism. In particular, W is Hausdorff.*

The natural injection $W \to \Gamma$ does *not* induce on W the topology from this lemma. Nevertheless, in practice it is the topology in this lemma that will be useful to us.

Proof. For any $w \in W$, we declare a base of opens around w to be given by sets of the form wU with U an open in I around the identity. In order to see that these satisfy the requirements to be the base for a topology on W, we must prove that if $x \in wU \cap w'U'$ in W then $xV \subset wU \cap w'U'$ for some open V in I around the identity. Since Γ is a topological group and I is being considered with its induced topology from Γ, we may multiply on the left by w^{-1} to reduce to the case $w = 1$, so $x = w'u' \in U \subseteq I$ with $u' \in U'$. Let $U = I \cap T$ and $U' = I \cap T'$ for opens T, T' in Γ. Then $V = I \cap x^{-1}T \cap x^{-1}w'T'$ works.

Now that we have a well-defined topology on W, we note that by the very definition I is an open subset which inherits its original topology. In order to prove the continuity of the group structure, we just have to prove continuity of multiplication at $(1,1)$ and of inversion at 1. As these are local questions, they may be checked upon restriction to an open neighborhood, such as $I \times I$ and I respectively. Since I inherits its original topology, and as such is a topological group (!), the desired continuity follows. The openness condition clearly makes this the unique such structure of topological group on W.

Finally, let $\varphi \in W$ project to $1 \in \mathbf{Z}$. We want to prove that the map $I \times \mathbf{Z} \to W$ defined by $(i,n) \mapsto i\varphi^n$ is a homeomorphism. Since W is a topological group, this map is certainly continuous. It is visibly bijective. Finally, to prove continuity of the inverse we note that

$$w \mapsto (w\varphi^{-\pi(w)}, \pi(w))$$

defines the inverse, where $\pi : W \to \mathbf{Z}$ is the projection. The map π has *open* kernel and hence is *continuous*, so we get continuity of this inverse.

■

Let us now see why the topology just defined on W does *not* coincide with the induced topology on W from Γ. For later purposes, we prove something a little stronger:

Lemma 3.2. *The natural injective of groups $\iota : W \to \Gamma$ is continuous with dense image, but I is not open in W with respect to this induced subspace topology.*

Proof. Continuity is a local question near the identity on W, so we can restrict attention to the open neighborhood I of the identity in W. But I as a subspace of W inherits its given topology from Γ, so continuity follows.

Now we check denseness. Let $\gamma \in \Gamma$ be an element. Let U be an open set in Γ around the identity. For denseness we must show W meets γU. Note that the set U^{-1} of inverses of elements of U is also open. Since $\pi : \Gamma \to \widehat{\mathbf{Z}}$ is a *quotient* map and is a homomorphism of topological groups, it follows that $\pi(U^{-1})$ is an open subset of the topological group $\widehat{\mathbf{Z}}$. By denseness of $\mathbf{Z}$ in $\widehat{\mathbf{Z}}$, it follows that $\mathbf{Z}$ meets the open neighborhood $\pi(\gamma)\pi(U^{-1}) = \pi(\gamma U^{-1})$ around $\pi(\gamma)$. Pick some $m \in \mathbf{Z}$ of the form $m = \pi(\gamma u^{-1})$ with $u \in U$. By definition, $\gamma u^{-1} \in \pi^{-1}(\mathbf{Z}) = W$. Thus, we have $\gamma = wu$ with $w = \gamma u^{-1} \in W$ and $u \in U$.

A similar argument shows that I cannot be open in W with respect to the induced topology from Γ, as follows. We pick an open set U in Γ and must prove $U \cap W \neq I$. As we have just seen, $\pi(U)$ is open in $\widehat{\mathbf{Z}}$. Since it contains the identity and $\{m\widehat{\mathbf{Z}}\}$ is a base of opens in $\widehat{\mathbf{Z}}$ around the identity, there is a $u \in U$ with $\pi(u) = m \in \mathbf{Z}$ and $m \neq 0$. Thus, $u \in U \cap W$ but $u \notin I$ (since $m \neq 0$). Hence, $U \cap W \neq I$.

■

For technical reasons, it is convenient to observe that the Weil group construction behaves well for formation of quotients by closed normal subgroups:

Lemma 3.3. *Let $I_0 \subseteq I$ be a closed subgroup which is normal in W. Then I_0 is a closed normal subgroup in Γ and the quotient Γ/I_0 is profinite. Moreover, W/I_0 with its quotient topology is, in the evident manner, the Weil group obtained from the data of $\Gamma/I_0 \to \widehat{\mathbf{Z}}$.*

Proof. Since conjugation by elements of W preserves I_0 and W is dense in the *Hausdorff topological* group Γ, it follows that conjugation by elements of Γ preserves I_0. That is, I_0 is normal in Γ. Being closed in I, which is turn is closed in Γ, we see that I_0 is closed in Γ. Any quotient of a profinite group by a normal closed subgroup is again profinite. Since I is open in W, by definition of the quotient topology we see that I/I_0 maps topologically isomorphically onto an open subgroup in the quotient group W/I_0. It remains to check that the natural continuous injective map $I/I_0 \to \Gamma/I_0$ induces the subspace topology on I/I_0. But I is profinite (being closed in the profinite Γ), so the quotients I/I_0 and Γ/I_0 are profinite. Thus, this injection is a continuous injection between profinite (hence compact Hausdorff) groups, so it is indeed a topological embedding.

■

4. The whole point of this

Here are the two fundamental basic properties of this Weil group construction:

Theorem 4.1. *Let $\iota : W \to \Gamma$ be the natural continuous injection. The operation $H \rightsquigarrow \iota^{-1}(H)$ sets up an inclusion-preserving bijection between open subgroups of finite index in Γ and open subgroups of finite index in W, with inverse given by $H' \rightsquigarrow \overline{\iota(H')}$. Moreover, the natural induced map of finite discrete coset spaces $W/\iota^{-1}(H) \to \Gamma/H$ is bijective for all such H. In particular, the profinite completion of W maps topologically isomorphically onto the profinite Γ.*

Proof. Let H be an open subgroup of finite index in Γ. By continuity, $\iota^{-1}(H)$ is an open subgroup in W. Moreover, the continuous map of discrete coset spaces $W/\iota^{-1}(H) \to \Gamma/H$ is injective, so $\iota^{-1}(H)$ has finite index in W. In order to see that this injective coset space map is bijective, we simply observe that for any $\gamma \in \Gamma$ the *non-empty open* set γH must meet the dense set $\iota(W)$. We now prove that $\overline{\iota^{-1}(H)} = H$. Since

- H is an *open* subgroup in Γ,
- Γ has a base of open subgroups around the identity,
- W is dense in Γ,

it follows that $\iota^{-1}(H) = W \cap H$ is dense in H. Since H is closed (as is any open subgroup in a topological group: the complement is a disjoint union of coset spaces all of which are open!), it follows that $\overline{\iota^{-1}(H)} = H$.

Finally, let H' be an open subgroup of finite index in W. We wish to prove that $H' = \iota^{-1}(H)$ for some open subgroup of finite index in Γ. Any finite index subgroup in an abstract group always contains a subgroup which is normal of finite index in the ambient group, and the construction is given by intersecting several conjugate subgroups. In particular, this construction preserves openness, so we conclude that H' contains a subgroup G' which is open of finite index in W and *normal*. If we can show $G' = \iota^{-1}(G)$ for some open subgroup of finite index in Γ, then by uniqueness (proven already!) such a G is stable under conjugation by W and hence by denseness of W in the *Hausdorff topological* Γ such a G is *normal* in Γ. The resulting isomorphism of finite *groups* $W/G' \simeq \Gamma/G$ then shows that the subgroup H'/G' on the left goes over to some H/G on the right, and this is the H we sought. Thus, we may assume that H' is also *normal* in W.

Let $H'' = H' \cap I$, a normal open subgroup in W. In particular, H'' is open in I, and hence closed in I. By Lemma 3.3, H'' is normal and closed in Γ and we may replace the data H', W, Γ with H'/H'', W/H'', Γ/H'' (note that H'/H'' *is* open of finite index in W/H''!). But H'' is *open* in the *compact* group I, so it has *finite* index in I. Thus, I/H'' is a finite group. That is, by passing to quotients by H'' we are reduced to the case where I is *finite* (and discrete) and $H' \cap I = \{1\}$. Let $H = \overline{H'}$, a closed subgroup in Γ. Since Γ is Hausdorff and H' is normal in the dense subgroup W, it follows that the closure H is normal in Γ. Also, H is clearly *compact* (even profinite).

We will prove that H is open of finite index in Γ and $H' = H \cap W$, thereby completing the proof. Since H' has finite index in W, its image in $\mathbf{Z}$ has finite index. This image must be $m\mathbf{Z}$ for some $m \in \mathbf{N}$. Then the continuous map $H \to \widehat{\mathbf{Z}}$ has image containing $m\mathbf{Z}$. But this image must be closed (as H is compact and $\widehat{\mathbf{Z}}$ is Hausdorff), so it must contain the closure $m\widehat{\mathbf{Z}}$ of $m\mathbf{Z}$ in $\widehat{\mathbf{Z}}$. Combining this with the fact that I is *finite* and $m\widehat{\mathbf{Z}}$ has finite index in $\widehat{\mathbf{Z}}$, it follows from the exact sequence (1.1) that H has finite index in Γ. But being closed of finite index, it must therefore be open!

Finally, we must show $H \cap W \subseteq H'$ (the reverse inclusion being clear). Certainly the preimage $\pi^{-1}(m\widehat{\mathbf{Z}})$ in Γ is closed and contains H' (by definition of m), so this preimage contains the closure H of H'. As the reverse containment was already established, we get $\pi(H) = m\widehat{\mathbf{Z}}$ and hence

$$m\mathbf{Z} = \pi(H') \subseteq \pi(H \cap W) \subseteq m\widehat{\mathbf{Z}} \cap \mathbf{Z} = m\mathbf{Z}.$$

Thus, $\pi(H \cap W) = \pi(H')$. To show that the inclusion $H' \subseteq H \cap W$ is an equality, it therefore remains to prove that both sides have the same intersection with $\ker(\pi) = I$.

That is, we want $\overline{H'} \cap I = H' \cap I$. This latter intersection is trivial, so we will prove $\overline{H'} \cap I = \{1\}$. Note that $H' \to \Gamma/I \simeq \widehat{\mathbf{Z}}$ is an injective homomorphism onto $m\mathbf{Z}$. As I is finite, to prove $\overline{H'} \cap I = \{1\}$ we just have to show that $\overline{H'}$ is torsion-free. The composite map

$$\mathbf{Z} \simeq m\mathbf{Z} \simeq H' \to \Gamma$$

to a *profinite* group uniquely extends to a continuous homomorphism

$$h : \widehat{\mathbf{Z}} \to \Gamma.$$

Moreover, $h(\mathbf{Z}) = H' \subseteq \overline{H'}$ and $\mathbf{Z}$ is dense in $\widehat{\mathbf{Z}}$, so h has image inside of $\overline{H'}$. But the image of h is compact, hence closed, and it contains H', so $h(\widehat{\mathbf{Z}}) = \overline{H'}$. Since $\widehat{\mathbf{Z}}$ is torsion-free, it suffices to prove that h is injective (and thus an isomorphism of groups from $\widehat{\mathbf{Z}}$ to $\overline{H'}$). The continuous composite $\pi \circ h : \widehat{\mathbf{Z}} \to \Gamma \twoheadrightarrow \widehat{\mathbf{Z}}$ is multiplication by m because this is true on the dense subset $\mathbf{Z}$. But $\widehat{\mathbf{Z}}$ is torsion-free, so $\pi \circ h$ is injective and thus h is injective as desired.

■

The second important property of the Weil group construction is that it commutes with formation of abelianizations! To be precise, using (1.2) we can make a Weil group construction for $\pi^{\mathrm{ab}} : \Gamma^{\mathrm{ab}} \twoheadrightarrow \widehat{\mathbf{Z}}$. Let us temporarily denote this $W_{\pi^{\mathrm{ab}}}$, so $W_{\pi^{\mathrm{ab}}}$ is topological group with dense image in Γ^{ab} and it contains the profinite $K = \ker(\pi^{\mathrm{ab}})$ as an open subgroup. The compatibility is:

Theorem 4.2. *The naturally induced continuous map $W^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ is injective with image equal to $W_{\pi^{\mathrm{ab}}}$, and the induced map of groups $W^{\mathrm{ab}} \to W_{\pi^{\mathrm{ab}}}$ is a topological isomorphism.*

One should appreciate the fact that the "abelianization" functor on topological groups does not usually preserve the property of being injective. The Langlands program is formulated in terms of representations of Weil(-Deligne) groups whereas class field theory provides a description of the image of the Weil group sitting inside of the abelianized Galois group (in the local non-archimedean and function field cases).

Proof. It is obvious that W, and hence W^{ab}, maps continuously *onto* the preimage $W_{\pi^{\mathrm{ab}}}$ of $\mathbf{Z}$ under π^{ab}. Grant for a moment that $W^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ is injective, and let us deduce the rest. It follows that we have a bijective continuous homomorphism of groups $j : W^{\mathrm{ab}} \to W_{\pi^{\mathrm{ab}}}$, compatible with the continuous map from each onto the discrete $\mathbf{Z}$. This map j then induces a continuous bijection between the respective *open* kernels of the projections to $\mathbf{Z}$. The continuity of j^{-1} is a local question near the identity, so it suffices to prove that the continuous bijection j induces a homeomorphism between the *open* kernels of the projections to $\mathbf{Z}$. As long as both kernels are *profinite*, such a continuous bijection between them is automatically a homeomorphism. By construction, the kernel of $W_{\pi^{\mathrm{ab}}} \to \mathbf{Z}$ is profinite. Meanwhile, the kernel K_0 of $W^{\mathrm{ab}} \to \mathbf{Z}$ is the image of the continuous injection $I/W_c \to W/W_c = W^{\mathrm{ab}}$, where W_c is the closure of the commutator subgroup of W (this lies inside I since I is closed in W and $W/I = \mathbf{Z}$ is abelian). Since the quotient topology on I/W_c is profinite (as I is profinite and W_c is closed), it remains to appeal to the rather general (and easy) fact that if $G \to G'$ is a topological embedding of topological groups (such as $I \to W$) and $H \subseteq G$ is a subgroup which is normal in G' (such as $H = W_c$), then the natural continuous map of quotient spaces $G/H \to G'/H$ is a topological embedding.

We now must verify the hypothesis that the map $W^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ is actually *injective*. In other words, if Γ_c and W_c denote the closures of the respective commutator subgroups of Γ and W, we want $W \cap \Gamma_c = W_c$. Since $W \to \Gamma$ is continuous, so $W \cap \Gamma_c$ is closed in W and contains all commutators of W, we see $W_c \subseteq W \cap \Gamma_c$. Conversely, suppose $w \in W \cap \Gamma_c$ and pick an open U in I around the identity. We want wU to contain a commutator in W (so the closed W_c in W is dense in the closed subgroup $W \cap \Gamma_c$ of W, whence these groups coincide). We can write $U = I \cap V$ for an open set V in Γ around the identity. Since Γ_c is the closure of the commutator subgroup of Γ and $w \in \Gamma_c$, we conclude that wV contains some commutator γ in Γ, say a finite product (in some order) of terms of the form $\gamma_{i,1}\gamma_{i,2}\gamma_{i,1}^{-1}\gamma_{i,2}^{-1}$. Now all commutators of Γ lie in I (since $\Gamma/I = \widehat{\mathbf{Z}}$ is abelian), so $\Gamma_c \subseteq I$. Thus, $w, \gamma \in I$. Writing $\gamma = wv$ with $v \in V$, we see $v = w^{-1}\gamma \in I$. If there exists a commutator γ' of W (rather than just one of Γ) such that $v' \overset{\mathrm{def}}{=} w^{-1}\gamma' \in V$, then $v' = V \cap I = U$, so wU contains a commutator γ' of W, as desired.

For *any* commutator γ' of W, we have

$$w^{-1}\gamma' = w^{-1}\gamma\gamma^{-1}\gamma'$$

and we recall that $w^{-1}\gamma$ is in the *open* set V of Γ. Since Γ is a topological group and $w^{-1}\gamma \in V$, it follows that for some small open neighborhood T around the identity in Γ we have $w^{-1}\gamma T \subseteq V$. Thus, it is enough to find a commutator γ' of W such that $\gamma^{-1}\gamma' \in T$, where T is a specified open around the identity in Γ and γ is a *fixed* commutator of Γ as above (in terms of $\gamma_{i,j}$'s). But Γ is a *topological* group and W is *dense* in Γ, so by approximating each of the finitely many $\gamma_{i,j}$'s by an element of W we can ensure the "approximating" commutator of W is inside of the small open neighborhood γT around the given commutator γ of Γ.

∎