Math 249B. Idele class groups and admissible homomorphisms

Let G be a Hausdorff topological group, and assume that it is *complete* in the sense that all Cauchy sequences in G converge; recall that a sequence $\{g_n\}$ in a Hausdorff topological group is called *Cauchy* if for every neighborhood N of 1 we have $g_n g_m^{-1} \in N$ for sufficiently large n and m (any convergent sequence is obviously Cauchy). For example, any compact Hausdorff group is complete since compact Hausdorff spaces are sequentially compact and a Cauchy sequence in the sense of the above definition is convergent if and only if it has a convergent subsequence.

Let K be a global field, let S be a finite set of places of K containing the archimedean places, and let I^S denote the free abelian group generated by the set of places not in S. We will denote this group additively, but you may prefer to think about it multiplicatively as the group of fractional ideals of $\mathscr{O}_{K,S}$ (except when S is empty in the function field case). A homomorphism $\phi : I^S \to G$ "is" the assignment of pairwise commuting elements $g_v \in G$ for each place $v \notin S$. Our goal here is to set up a bijection between certain homomorphisms $I^S \to G$ and certain homomorphisms $\mathbf{A}_K^\times \to G$.

1. Admissible homomorphisms

With S fixed as above, we define the *S-adele ring* $\mathbf{A}_K^S$ like $\mathbf{A}_K$ except that we omit the places in S. This has a locally compact Hausdorff topology and its unit group $\mathbf{A}_K^{S,\times}$ is topologized just like the ideles. For non-empty S, the classical *strong approximation theorem* (for the additive group) as in Chapter II of the Cassels-Fröhlich book says that the image of K in $\mathbf{A}_K^{S,\times}$ is dense; we will not use this fact.

We define the locally compact Hausdorff ring $K_S = \prod_{v\in S} K_v$, so clearly $\mathbf{A}_K = K_S \times \mathbf{A}_K^S$ as topological rings and hence likewise $\mathbf{A}_K^\times = K_S^\times \times \mathbf{A}_K^{S,\times}$ as topological groups (where $K_S^\times = \prod_{v\in S} K_v^\times$). Corresponding to this decomposition of $\mathbf{A}_K^\times$ we write $\alpha = \{\alpha\}_S \cdot \{\alpha\}^S$ for an idele α, so $\{\alpha\}_S$ and $\{\alpha\}^S$ may be viewed as the respective projections of α onto the "S part" and "away-from-S part" of the idele group. Alternatively, we view $\{\alpha\}_S$ as an idele whose components at factors away from S are all equal to 1 and we view $\{\alpha\}^S$ as an idele whose components at factors in S are all equal to 1. We use both points of view without comment.

We define the surjective homomorphism $(\cdot)^S : \mathbf{A}_K^\times \twoheadrightarrow I^S$ by $(\alpha)^S = \sum_{v\notin S} \mathrm{ord}_v(\alpha_v)v$. Clearly $(\alpha)^S = (\{\alpha\}^S)^S$, and the kernel of $(\cdot)^S$ is the open subgroup $K_S^\times \times \prod_{v\in S} \mathscr{O}_v^\times$ in $\mathbf{A}_K^\times$, so the quotient topology makes I^S discrete (as it should be).

Definition 1.1. A homomorphism $\phi : I^S \to G$ to a complete Hausdorff topological group is *admissible* if, for each neighborhood N of the identity in G, there exists an open neighborhood $U \subseteq K_S^\times$ around the identity such that $\phi((a)^S) \in N$ for all $a \in K^\times$ satisfying $\{a\}_S \in U$.

Equivalently, for all neighborhoods N around the identity in G, there exists $\varepsilon > 0$ such that if $a \in K^\times$ satisfies $|a-1|_v < \varepsilon$ for all $v \in S$ then $\phi((a)^S) \in N$.

This definition could be made without a completeness condition on G, but it does not seem to be of much interest in such generality. An alternative is to take U above to be a subgroup rather than just a neighborhood:

Definition 1.2. A homomorphism $\phi : I^S \to G$ to a complete Hausdorff topological group is *group-admissible* if, for each neighborhood N of the identity in G, there exists an open subgroup $U \subseteq K_S^\times$ around the identity such that $\phi((a)^S) \in N$ for all $a \in K^\times$ satisfying $\{a\}_S \in U$.

Equivalently, for all neighborhoods N around the identity in G, there exists $\varepsilon > 0$ such that if $a \in K^\times$ satisfies $|a-1|_v < \varepsilon$ for all non-archimedean $v \in S$ and $a > 0$ in $K_v^\times$ for all real v then $\phi((a)^S) \in N$.

The reason for the equivalence in the definition of group-admissibility is that any for non-archimedean v the space $K_v^\times$ has a base of open neighborhoods around 1 given by open subgroups $1+\mathfrak{m}_v^e$ for $e \geq 1$, and that the only open subgroups of $\mathbf{R}^\times$ (resp. $\mathbf{C}^\times$) are $\mathbf{R}^\times$ and $\mathbf{R}_{>0}^\times$ (resp. $\mathbf{C}^\times$) due to *connectivity* of $\mathbf{R}_{>0}^\times$ and $\mathbf{C}^\times$ and the fact that an open subgroup of a topological group is necessarily closed (as its complement is a union of necessarily open cosets).

The definition of admissible as given above is what is used in Cassels-Fröhlich. In general, group-admissibility has the effect if imposing much stronger conditions on ϕ at the archimedean places, so for global function fields it coincides with admissibility for but number fields it seems to be a much stronger property on ϕ in general. However, for profinite G (the case of most immediate interest in class field theory!) it turns out that if ϕ is admissible then it is necessarily group-admissibile. The proof rests on weak approximation and the completeness of the target, and it seems simplest to give the proof of the equivalence after we set up some relations between admissible ϕ's (in the sense of the above definition) and certain idele class characters. Thus, at the end of these notes we will return to the equivalence between the two definitions when the target G is a profinite group.

Remark 1.3. In Exercise 1 of HW3 generalized ideal class groups were related to quotients of I^S (for varying S) modulo subgroups of elements $(a)^S$ for $a \in K^\times$ satisfying suitable congruence conditions at the non-archimedean places in S and positivity conditions at some (possible empty) set of real places. From this point of view, it follows from the equivalence of admissibility and group-admissibility for profinite targets that admissible homomorphisms to finite (discrete) G are the "same" as homomorphisms to G from the generalized ideal class group for some modulus.

2. Hecke homomorphisms

Let G be a complete Hausdorff topological group. A *Hecke homomorphism* for K with values in G is a continuous homomorphism $\psi : \mathbf{A}_K^\times \to G$ such that $\psi(K^\times) = 1$, so equivalently it is a continuous homomorphism $\mathbf{A}_K^\times/K^\times \to G$. The most classical case is $G = \mathbf{C}^\times$, in which case ψ is traditionally called a *Hecke character* (or *grossencharacter*). (The terminology "Hecke homomorphism" is inspired by this special case, and is artifically made up for these notes; it is not standard.)

We are going to set up a bijection between Hecke homomorphisms for K with values in G and admissible homomorphisms from I^S to G up to a mild equivalence relation.

Definition 2.1. Let S and S' be two finite sets of places of K containing the archimedean places. Homomorphisms $\phi : I^S \to G$ and $\phi' : I^{S'} \to G$ are *equivalent* if there exists a finite set of places S'' containing both S and S' such that ϕ and ϕ' coincide on $I^{S''}$ (viewed as a subgroup of both I^S and $I^{S''}$).

The class of complete G's of most interest to use here will be:

Definition 2.2. A topological group G *has no small subgroups* if there is a neighborhood of 1 in G that contains no non-trivial subgroups of G.

A profinite group has no small subgroups if and only if it is finite. However, there are many interesting examples of complete Hausdorff topological groups that have no small subgroups. A simple example of much importance is $G = \mathbf{C}^\times$ (in which case the "no small subgroups" property is a simple exercise with polar form of complex numbers; see HW7). Rather generally, it is a basic exercise in Lie theory that *any* Lie group has no small subgroups.

Theorem 2.3. *Let G be a complete Hausdorff topological group.*

- *If $\phi : I^S \to G$ is an admissible homomorphism then there exists a unique Hecke homomorphism $\psi : \mathbf{A}_K^\times \to G$ such that $\psi(\alpha) = \phi((\alpha)^S)$ for all $\alpha \in \mathbf{A}_K^{S,\times}$. Moreover, ψ only depends on the equivalence class of ϕ, and $\phi \mapsto \psi$ defines an injection from the set of equivalence classes of admissible homomorphisms (with variable S) to the set of Hecke homomorphisms for K with values in G.*
- *If G has no small subgroups, then the injection just constructed is surjective: every Hecke homomorphism $\psi : \mathbf{A}_K^\times \to G$ arises from an admissible homomorphism $\phi : I^S \to G$ for some S.*

Proof. Let us first proof uniqueness in (1). By hypothesis, for any $x \in \mathbf{A}_K^{S,\times}$ we have $\psi(x) = \phi((x)^S)$. Thus, for any $x \in \mathbf{A}_K^\times$ we have $\psi(\{x\}^S) = \phi((\{x\}^S)^S) = \phi((x)^S)$, so for any $\alpha \in \mathbf{A}_K^\times$ and $a \in K^\times$ we have

$$\psi(\alpha) = \psi(a\alpha) = \psi(\{a\alpha\}_S)\psi(\{a\alpha\}^S) = \psi(\{a\alpha\}_S)\phi((a\alpha)^S).$$

Now use weak approximation to find $\{a_n\}$ in $K^\times$ such that $a_n \to \alpha_v^{-1}$ in $K_v^\times$ for all $v \in S$. Thus, $\{a_n\alpha\}_S \to 1$ in $K_S^\times$, so by continuity of ψ, it follows that $\psi(\{a_n\alpha\}_S) \to 1$ in G. This gives that the sequence of points $\phi((a_n\alpha)^S) = \psi(\{a_n\alpha\}_S)^{-1}\psi(\alpha)$ admits a limit as $n \to \infty$, namely $\psi(\alpha)$. The resulting formula

$$\psi(\alpha) = \lim_{n\to\infty} \phi((a_n\alpha)^S)$$

uniquely determines ψ, since the right side has nothing to do with ψ. Note that this part of the proof did not use admissibility of ϕ, but it also does not address the important problem of existence for ψ.

Before we prove existence in (1), let us use uniqueness to prove that ψ only depends on the equivalence class of ϕ. In view of how equivalence is defined, it suffices to show that if S' contains S and ϕ' is the restriction of ϕ to $I^{S'}$ then the Hecke homomorphism ψ linked to ϕ also "works" for ϕ'. That is, for $\alpha \in \mathbf{A}_K^{S',\times}$ we want $\psi(\alpha) = \phi'((\alpha)^{S'})$. Since α also lies in $\mathbf{A}_K^{S,\times}$, it is equivalent to show $\phi((\alpha)^S) = \phi((\alpha)^{S'})$. But obviously $(\alpha)^S = (\alpha)^{S'}$ (via the inclusion of $I^{S'}$ into I^S) since $\alpha \in \mathbf{A}_K^{S,\times}$, so we get the desired result.

It remains for (1) to prove existence, and we wish to take the limit formula as our definition: for any $\alpha \in \mathbf{A}_K^\times$, we choose $\{a_n\}$ in $K^\times$ such that $a_n \to \alpha_v^{-1}$ in $K_v^\times$ for all $v \in S$, and we define

$$\psi(\alpha) = \lim_{n\to\infty} \phi((a_n\alpha)^S)$$

in G. Of course, we must prove that this limit exists and that it is independent of the choice of a_n's; it then will be proved that ψ is a Hecke homomorphism satisfying the required properties. Since G is complete, for existence of the limit it suffices to prove that the sequence of points $\phi((a_n\alpha)^S)$ is Cauchy in G. For any $n, m \geq 1$, we have

$$\phi((a_n\alpha)^S)\phi((a_m\alpha)^S)^{-1} = \phi((a_n/a_m)^S),$$

and for each $v \in S$ the sequence of a_n's is Cauchy in $K_v^\times$ because it has a limit (namely, α_v^{-1}). Thus, for any neighborhood of 1 in $K_S^\times$ we see that the point $\{a_n/a_m\}_S \in K_S^\times$ lies in this neighborhood for sufficiently large n and m. The admissibility of ϕ therefore implies that for any neighborhood N of 1 in G we have $\phi((a_n/a_m)^S) \in N$ for sufficiently large n and m. This verifies the existence of the desired limit in G.

If $\{a'_n\}$ is another sequence in $K^\times$ whose components at each $v \in S$ converge to α_v^{-1} in $K_v^\times$, then we need to prove that the convergent sequence of $\phi((a'_n\alpha)^S)$'s has the same limit as the convergent sequence of $\phi((a_n\alpha)^S)$'s. Equivalently, we want the points $\phi((a'_n\alpha)^S)\phi((a_n\alpha)^S)^{-1} \in G$ to converge to 1 as $n \to \infty$. The nth such point is $\phi((a'_n/a_n)^S)$, so since $a'_n/a_n \to \alpha_v^{-1}/\alpha_v^{-1} = 1$ in $K_v^\times$ for each $v \in S$, we may use the admissibility of ϕ as above to conclude that $\phi((a'_n/a_n)^S) \to 1$. This completes the verification that the definition of $\psi(\alpha)$ makes sense and only depends on α.

Since the definition of $\psi(\alpha)$ does not depend on the specific choice of a_n's, it is clear that ψ is a homomorphism. Also, if $\alpha = c \in K^\times$ then we can take $a_n = c^{-1}$ for all n and hence $\psi(\alpha) = 1$. That is, $\psi(K^\times) = 1$. If instead $\alpha \in \mathbf{A}_K^{S,\times}$ then we can take $a_n = 1$ for all n, and so in this case we get the desired identity $\psi(\alpha) = \phi((\alpha)^S)$. It remains to prove that ψ is continuous (and hence a Hecke homomorphism). So far, the arguments would have worked if we had required ϕ to be group-admissible instead of admissible. To prove the continuity of ψ, we shall require admissibility.

Since ψ is a homomorphism between topological groups, to prove it is continuous we only need to check continuity at the identity. Since $\mathbf{A}_K^\times$ has a countable base of opens around the identity, we can check continuity using sequences: if $\{\alpha_m\}$ in $\mathbf{A}_K^\times$ tends to 1 then we want $\psi(\alpha_m)$ to converge to 1 in G. Since $\alpha_m \to 1$ in $\mathbf{A}_K^\times$, for large m we have

$$\alpha_m \in K_S^\times \times \prod_{v \notin S} \mathscr{O}_v^\times = \ker((\cdot)^S).$$

For each m, let $\{a_n(\alpha_m)\}_{n \geq 1}$ be a sequence in $K^\times$ as in the definition of $\psi(\alpha_m)$. For large m and any $n \geq 1$, we have

$$\phi((a_n(\alpha_m)\alpha_m)^S) = \phi((a_n(\alpha_m))^S)\phi((\alpha_m)^S) = \phi((a_n(\alpha_m))^S)$$

since $(\alpha_m)^S = 1$ for large m. Thus, $\psi(\alpha_m) = \lim_{n\to\infty} \phi((a_n(\alpha_m))^S)$. Let N be a neighborhood of 1 in G. By admissibility of ϕ, if $c \in K^\times$ is sufficiently close to 1 in $K_S^\times$ then $\phi((c)^S) \in N$. But taking m large makes the S-part $\{\alpha_m\}_S \in K_S^\times$ very close to 1 (since $\alpha_m \to 1$ in $\mathbf{A}_K^\times$), whence the points $\{\alpha_m^{-1}\}_S = \{\alpha_m\}_S^{-1}$ get very close to 1. For each m we have $\{a_n(\alpha_m)\}_S \to \{\alpha_m^{-1}\}_S$ in $K_S^\times$ as $n \to \infty$, so for each m we see that the S-parts $\{a_n(\alpha_m)\}_S$ get very close to 1 for n sufficiently large (depending on m!). Hence, provided m is large enough, $\phi((a_n(\alpha_m))^S) \in N$ when n is sufficiently large (depending on m). It follows that the limit $\psi(\alpha_m)$ lies in N for m very large, and so obviously the sequence of $\psi(\alpha_m)$'s tends to 1. This completes the proof of continuity of ψ, and so verifies (1).

Now we turn to the proof of (2). Let N be a neighborhood of 1 in G such that N has no non-trivial subgroups. A base of opens around the identity in $\mathbf{A}_K^\times$ is given by sets of the form

$$U_\Sigma = \prod_{v \in \Sigma} U_v \times \prod_{v \notin \Sigma} \mathscr{O}_v^\times$$

for finite sets of places Σ containing all archimedean places (this notation "U_Σ" is inherently ambiguous since it hides the specific small open U_v's around 1 being used in factors at $v \in S$, but this ambiguity should not lead to confusion). Since ψ is continuous, $\psi^{-1}(N)$ is a neighborhood of 1 in $\mathbf{A}_K^\times$ and hence $\psi^{-1}(N)$ contains U_S for some suitable S. In particular, $\psi^{-1}(N)$ contains $\prod_{v \notin S} \mathscr{O}_v^\times$ and thus N contains the subset $\psi(\prod_{v \notin S} \mathscr{O}_v^\times)$. But this latter subset of G is a subgroup since ψ is a homomorphism, so the condition on N implies that this subset is $\{1\}$. In other words, ψ kills $\prod_{v \notin S} \mathscr{O}_v^\times$. It follows that the restriction ψ^S of ψ to $\mathbf{A}_K^{S,\times}$ kills the "maximal compact" subgroup $\prod_{v \notin S} \mathscr{O}_v^\times$, and hence ψ^S factors through the quotient $I^S = \mathbf{A}_K^{S,\times} / \prod_{v \notin S} \mathscr{O}_v^\times$. The projection to this quotient is exactly the homomorphism $\alpha \mapsto (\alpha)^S$ for $\alpha \in \mathbf{A}_K^{S,\times}$, so we have built a homomorphism $\phi : I^S \to G$ such that $\psi(\alpha) = \phi((\alpha)^S)$ for all $\alpha \in \mathbf{A}_K^{S,\times}$.

We need to prove that ϕ is admissible. Let N' be a neighborhood of 1 in G, so N'^{-1} is also a neighborhood of 1 and thus the preimage $\psi^{-1}(N'^{-1})$ is a neighborhood of 1 in $\mathbf{A}_K^\times$, so it contains U_Σ for some Σ that we may enlarge if necessary to contain S. The neighborhood $\psi^{-1}(N'^{-1})$ has a crucial property: it is stable under multiplication by the subgroup $\prod_{v\notin S}\mathscr{O}_v^\times \in \psi^{-1}(1)$. Thus, it contains the subset

$$U_\Sigma \cdot \prod_{v\notin S}\mathscr{O}_v^\times$$

that in turn clearly contains U_S (say using the same small open subsets $U_v \subseteq K_v^\times$ at $v\in S\subseteq\Sigma$ that arise in U_Σ). If $c\in K^\times$ satisfies $\{c\}_S\in\prod_{v\in S}U_v$ inside $K_S^\times$ then we get $\psi(\{c\}_S)\in {N'}^{-1}$, so $\psi(\{c\}_S)^{-1}\in N'$. For such c we use the relation $\psi(\alpha)=\phi((\alpha)^S)$ for $\alpha\in\mathbf{A}_K^{S,\times}$ to deduce

$$1=\psi(c)=\psi(\{c\}_S)\psi(\{c\}^S)=\psi(\{c\}_S)\phi((\{c\}^S)^S)=\psi(\{c\}_S)\phi((c)^S),$$

so $\phi((c)^S)=\psi(\{c\}_S)^{-1}\in N'$ and hence ϕ is admissible. ■

Let us now return to the issue of proving that if G is profinite and $\phi: I^S\to G$ is an admissible homomorphism then it is group-admissible; that is, we claim that if N is a neighborhood of 1 in G and $a\in K^\times$ satisfies the properties that $a\in K_v^\times$ sufficiently close to 1 for each non-archimedean $v\in S$ and $a\in K_v^\times$ is positive for each real v, then necessarily $\phi((a)^S)\in N$. This is only an issue in the case of number fields, so we now take K to be a number field. It suffices to work with a cofinal system of N's, and so with profinite G it suffices to restrict attention to the case when N is an open normal subgroup of G. The condition on N in such cases is a vanishing condition for the composite of ϕ with the projection map $G\to G/N$, so we may rename G/N as G and thereby reduce the equivalence problem to the case of finite discrete G. In this case, we must prove that $\phi((a)^S)=1$ for $a\in K^\times$ such that $\{a\}_S$ has non-archimedean components sufficiently close to 1 and real components positive (in the respective factor fields K_v for $v\in S$).

By the main theorem above, there exists a Hecke homomorphism $\psi:\mathbf{A}_K^\times\to G$ such that $\psi(\alpha)=\phi((\alpha)^S)$ for all $\alpha\in\mathbf{A}_K^{S,\times}$. Since $\{a\}^S\in\mathbf{A}_K^{S,\times}$, we have $\psi(\{a\}^S)=\phi((\{a\}^S)^S)=\phi((a)^S)$ for all $a\in K^\times$. Also, since $\psi(a)=1$ (as ψ is a Hecke homomorphism) and $a=\{a\}_S\{a\}^S$ in $\mathbf{A}_K^\times$, $\psi(\{a\}^S)=\psi(\{a\}_S)^{-1}$. Our problem is therefore to prove that $\psi(\{a\}_S)=1$ for $a\in K^\times$ whose image in $K_v^\times$ is sufficiently close to 1 for non-archimedean $v\in S$ and is positive for real v. Letting $\psi_v=\psi|_{K_v^\times}$ for each place v of K, $\psi_v:K_v^\times\to G$ is a continuous homomorphism and for any $\alpha=(\alpha_v)\in\prod_{v\in S}K_v^\times=K_S^\times$ clearly $\psi(\alpha)=\prod_{v\in S}\psi_v(\alpha_v)$.

We are now reduced to the following problem: prove that for each non-archimedean place v of K the map ψ_v kills a sufficiently small neighborhood of 1 in $K_v^\times$ (only the *finitely many* non-archimedean $v\in S$ are of relevance to us), for each real place v the map ψ_v kills the subgroup of positive elements in $K_v^\times$, and for each complex place v the map ψ_v is trivial. Since G is a finite discrete group, the kernel of the continuous homomorphism $\psi_v: K_v^\times\to G$ contains a neighborhood of 1 and thus is an open subgroup. For non-archimedean v this kernel therefore contains a sufficiently small neighborhood of 1 and for real (resp. complex) v it contains $K_{v,>0}^\times$ (resp. $K_v^\times$) because the only open subgroup of $\mathbf{R}_{>0}^\times$ (resp. $\mathbf{C}^\times$) is the entire group. This completes the proof that for profinite G all admissible homomorphisms $\phi: I^S\to G$ are automatically group-admissible.

Since the classical formulation of the Artin reciprocity law says that for finite abelian L/K the Artin map $\mathrm{Art}_{L/K,S}: I^S\to\mathrm{Gal}(L/K)$ is admissible for some finite set of places S of K containing the archimedean places and those that are ramified in L, we conclude that this form of reciprocity law is equivalent to the statement that $\mathrm{Art}_{L/K,S}$ is group-admissible for sufficiently large S (depending on L/K). It is the property of group-admissibility that actually emerges from the classical proofs of global class field theory, and it is important when formulating the main results of the classical theory in terms of generalized ideal class groups.