Math 249B. L-functions and Dirichlet density for global fields

1. Motivation

Dirichlet introduced the functions $L(s,\chi)$ for Dirichlet characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ in order to study primes in arithmetic progressions. More specifically, he defined a notion of *Dirichlet density* for an arbitrary set of primes, and he proved that for each congruence class c in $(\mathbf{Z}/N\mathbf{Z})^\times$ the set of positive primes p such that $p \bmod N = c$ has a Dirichlet density and it is equal to $1/\phi(N) = 1/|(\mathbf{Z}/N\mathbf{Z})^\times|$. Roughly speaking, primes are "equally distributed" across congruence classes mod N, at least in the sense of Dirichlet density. In this classical setting, the Dirichlet density of a set Σ of positive primes is

$$\delta_{\mathrm{Dir}}(\Sigma) \stackrel{\mathrm{def}}{=} \lim_{s\to 1^+} \frac{\sum_{p\in\Sigma} p^{-s}}{\sum_p p^{-s}} = \lim_{s\to 1^+} \frac{\sum_{p\in\Sigma} p^{-s}}{\log(1/(s-1))}$$

if this limit exists; the final equality rests on the fact that $\zeta(s) \sim 1/(s-1)$ as $s \to 1^+$, which implies

$$\log(1/(s-1)) \sim \log\zeta(s) = \sum_p -\log(1-p^{-s}) \sim \sum_p p^{-s}$$

as $s \to 1$ because $\log(1/(s-1)) \to \infty$ yet upon expanding out $-\log(1-p^{-s}) = \sum_{j\geq 1} p^{-sj} = p^{-s} + \sum_{j\geq 2} p^{-sj}$ we see that the contribution $\sum_p \sum_{j\geq 2} p^{-sj}$ is bounded by $\zeta_{\mathbf{Q}}(2s)$ for real s near 1 (and even for real $s > 1/2$).

There is another notion of density that comes to mind, *natural density*:

$$\delta_{\mathrm{nat}}(\Sigma) \stackrel{\mathrm{def}}{=} \lim_{x\to\infty} \frac{\#\{p \leq x \,|\, p \in \Sigma\}}{\#\{p \leq x\}} = \frac{\#\{p \leq x \,|\, p \in \Sigma\}}{x/\log(x)},$$

using the prime number theorem for the final equality. It is a general fact that if Σ admits a natural density then it admits a Dirichlet density and these coincide. However, the converse is false: there are examples of sets of primes that admit a Dirichlet density but not a natural density; in this sense, Dirichlet density is a strictly more general notion (if perhaps a bit less intuitive than natural density). Dirichet's theorem is true in the sense of natural density, and this is a genuinely stronger assertion than the traditional form in terms of Dirichlet density. The main difference between the two concepts is that natural density rests on an ordering of the set of primes, whereas Dirichlet density does not.

There are two ways to generalize Dirichlet's theorem. One method, which I believe came historically first, arose from the question of equidistribution of primes in class groups of number fields K. To make the link more substantive, one should really work with generalized ideal class groups $\mathrm{Cl}_{\mathfrak{m}}(K)$, as then for $K = \mathbf{Q}$ and $\mathfrak{m} = N\infty$ this recovers $(\mathbf{Z}/N\mathbf{Z})^\times$ as used by Dirichlet. Presumably with this motivation in mind, Weber viewed characters $\psi : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbf{C}^\times$ as a natural generalization of Dirichlet characters (these were called *ideal class characters*) and he defined an L-function

$$L_W(s,\psi) = \prod_{v\nmid\mathfrak{m}\infty} (1-\psi(\mathfrak{p}_v)q_v^{-s})^{-1}$$

for $\mathrm{Re}(s) > 1$ (where q_v denotes the size of the residue field at v), with an eye toward proving good analytic and non-vanishing properties so as to deduce an equidistribution result for prime ideals in generalized ideal class groups.

In a completely different direction, Artin viewed $(\mathbf{Z}/N\mathbf{Z})^\times$ not as a generalized ideal class group but rather as a Galois group: $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$. (This was after Takagi had proved class field theory in a crude form, so Artin knew that the two viewpoints of abelian Galois groups and generalized ideal class groups were closely related.) For a positive prime p, the canonical identification of $(\mathbf{Z}/N\mathbf{Z})^\times$ with $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ carries $p \bmod N$ to the Frobenius element for $p\mathbf{Z}$. Thus, Artin envisioned a generalization of Dirichlet's theorem going way beyond the classical case by allowing even *non-abelian* Galois groups and considering equidistribution for conjugacy classes in Galois groups as Frobenius conjugacy classes. To be more precise, if K'/K is a finite Galois extension of global fields then all but finitely many non-archimedean places v of K are unramified in K' and so the Frobenius elements $\mathrm{Frob}(v'|v) \in G = \mathrm{Gal}(K'/K)$ make sense for places v' on K' over v. As we vary v', these Frobenius elements sweep out a conjugacy class in G, the *Frobenius* *conjugacy class of* v in G. (If G is abelian then conjugacy classes are elements and so it is well-posed to speak of a Frobenius element attached to a finite place v of K that is unramified in K'.) Hence, one is led to ask if a given conjugacy class c of G is a Frobenius conjugacy class for some non-archimedean place v of K that is unramified in K', and if so then "how often"? For example, can we say in a precise sense that the proportion of unramified v for which c is the Frobenius conjugacy class of v is $|c|/|G|$? This would be a good version of an "equidistribution" result for Frobenius classes in G as we vary v on K.

To make a precise statement of equidistribution along the lines considered by either Weber or Artin, it is necessary to first define a notion of *Dirichlet density* for a set of places of K. Our aim in this handout is to (i) define and analyze convergence properties for the relevant L-functions in the "abelian" setting, (ii) define Dirichlet density in the broader context of global fields (not just $\mathbf{Q}$ as for Dirichlet), (iii) define and summarize basic properties of Artin L-functions which go beyond the abelian setting, and finally (iv) discuss the important Chebotarev density theorem that vastly generalizes Dirichlet's theorem on primes in arithmetic progressions (and via class field theory recovers as a special case the equidistribution result one wants for prime ideal classes in generalized ideal class groups). The modern proof of Chebotarev's theorem rests on class field theory, though historically Chebotarev proved his result without class field theory and his theorem played a pivotal role in motivating some aspects of Artin's work on class field theory (following Takagi's work).

2. Abelian L-functions

For both historical and technical reasons, we begin our discussion of "abelian" L-functions by focusing on the very special case of ζ-functions (which correspond to the trivial character in the general case). Let K be a global field. The *ζ-function* of K is defined by

$$\zeta_K(s) = \prod_{v\nmid\infty}(1-q_v^{-s})^{-1}$$

for $\mathrm{Re}(s) > 1$, where q_v denotes the size of the residue field at v. If K is a number field, then formally expanding out the geometric series as in the classical case $K = \mathbf{Q}$ gives the alternative form $\zeta_K(s) = \sum_{\mathfrak{a}\neq 0}\mathbf{N}(\mathfrak{a})^{-s}$, up to checking convergence issues. In the function field case we get a similar expression, except the sum is taken over the set $\mathrm{Div}^+(X)$ of all nonzero effective divisors D on the corresponding smooth projective algebraic curve X over the field of constants k, and the ideal-norm is replaced with $q^{\deg_k(D)}$.

We wish to check that the above manipulations make sense and behave exactly as in the classical case for $K = \mathbf{Q}$. That is, both the sum and product should be absolutely and uniformly convergent in any closed half-plane $\mathrm{Re}(s) \geq 1+\varepsilon$ with $\varepsilon > 0$ (so we have analyticity and non-vanishing for $\mathrm{Re}(s) > 1$). To do this, we exploit a trick to reduce problems to the basic cases $\mathbf{Q}$ and $\mathbf{F}_p(t)$ where everything can be seen by hand: we view K as a finite separable extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$ (so in the function field case we are picking a separating transcendence basis for K over $\mathbf{F}_p$, as may always be found since $\mathbf{F}_p$ is perfect). To thereby reduce our problem for K to the special case of $\mathbf{Q}$ or $\mathbf{F}_p(t)$, more generally suppose that K/K_0 is a finite separable extension of some degree $d \geq 1$. Then each place v of K_0 lifts to at most d places w of K, and $q_w \geq q_v$. Hence, for $s > 0$ we have $\prod_{w|v}(1-q_w^{-s})^{-1} \leq (1-q_v^{-s})^{-d}$. Elementary estimates akin to the classical treatment for $K = \mathbf{Q}$ then show that $\zeta_K(s) \leq \zeta_{K_0}(s)^d$, and more specifically that to prove our convergence and uniformity claims with K it is enough to work with K_0. (Note that separability of K/K_0 was not actually needed in this argument.)

The argument for $K = \mathbf{Q}$ is classical, by shifting to the summation expression rather than the Euler product for $\zeta_{\mathbf{Q}}$ and doing a comparison with $\int_1^\infty x^{-s}\,\mathrm{d}x$ for $s > 1$. The case $K = \mathbf{F}_p(t)$ turns out to be easier in the sense that no integral comparison is necessary once we expand out the sum: the summation expression is

$$\zeta_{\mathbf{F}_p(t)}(s) = \sum_{D\in\mathrm{Div}^+(\mathbf{P}^1_{\mathbf{F}_p})} p^{-s\deg_{\mathbf{F}_p}(D)} = \sum_{r\geq 1} d_p(r)p^{-sr}$$

with $d_p(r)$ denoting the number of D with $\deg_{\mathbf{F}_p}(D) = r$. Accounting for the possible support at the $\mathbf{F}_p$-rational point at infinity (where the coefficient is between 0 and r), we have $d_p(r) = \sum_{0\le j\le r} \nu_p(j)$ where $\nu_p(j)$ is the number of D supported away from ∞ and satisfying $\deg_{\mathbf{F}_p}(D) = j$. In other words, $\nu_p(j)$ is the number of monic polynomials over $\mathbf{F}_p$ with degree j, which is to say $\nu_p(j) = p^j$, so $d_p(r) = 1+p+\cdots+p^r \le (p/(p-1))p^r$. Dropping the constant factor of $p/(p-1)$ in this upper bound, it suffices to check the convergence of $\sum_{r\ge1} p^r p^{-(1+\varepsilon)r} = \sum_{r\ge1} p^{-\varepsilon r}$, which is obvious.

Having treated the case of ζ-functions, we can now take up the general case. First we define Weber L-functions relative to a modulus.

Definition 2.1. If $\mathfrak{m}$ is a modulus for a global field K and $\psi : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbf{C}^\times$ is a character then the associated *Weber L-function* is the Euler product

$$L_{W,\mathfrak{m}}(s,\psi) = \prod_{v\nmid\mathfrak{m}\infty} (1-\psi([\mathfrak{p}_v])q_v^{-s})^{-1}$$

for $\mathrm{Re}(s) > 1$.

When $\mathfrak{m}$ is understood from context, we may just write $L_W(s,\psi)$. (A simple example of possible confusion already occurs for $K = \mathbf{Q}$ due to the equality $(\mathbf{Z}/6\mathbf{Z})^\times = (\mathbf{Z}/3\mathbf{Z})^\times$. This is why we have to be aware of the modulus we are using.) There is a fundamental defect of the theory of Weber L-functions, which generalizes a defect of the theory of Dirichlet L-functions: the artifice of "imprimitive" characters. More specifically, for any character $\psi : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbf{C}^\times$ we can consider whether it may factor through some (necessarily unique) $\psi' : \mathrm{Cl}_{\mathfrak{m}'}(K) \to \mathbf{C}^\times$ via the projection $\mathrm{Cl}_{\mathfrak{m}}(K) \twoheadrightarrow \mathrm{Cl}_{\mathfrak{m}'}(K)$ for a modulus $\mathfrak{m}'|\mathfrak{m}$. Using the identification

$$\mathrm{Cl}_{\mathfrak{m}}(K) \simeq \mathbf{A}_K^\times/K^\times U_{\mathfrak{m}}$$

that respects change in $\mathfrak{m}$, we see that among all such $\mathfrak{m}'$ for a given ψ there is a unique minimal one $\mathfrak{f}_\psi$ which divides all others. This is called the *Weber conductor* of ψ, and we call ψ *primitive* (at level $\mathfrak{m}$) if $\mathfrak{m} = \mathfrak{f}_\psi$; otherwise ψ is *imprimitive*. In the special case $K = \mathbf{Q}$ this essentially recovers the notion of primitive versus imprimitive Dirichlet characters at a given modulus, up to the fact that now we are keeping track of the real place (which is ignored in the definition of $L_{W,\mathfrak{m}}(s,\psi)$ and so does not matter here, but will come back to haunt us when figuring out the right Γ-factors at the infinite places needed to have a good functional equation).

The key point is that if $\psi : \mathrm{Cl}_{\mathfrak{m}}(K) \to \mathbf{C}^\times$ is a character with conductor $\mathfrak{f}$ and $\psi' : \mathrm{Cl}_{\mathfrak{f}}(K) \to \mathbf{C}^\times$ is the corresponding primitive character then $L_{W,\mathfrak{m}}(s,\psi)$ differs from $L_{W,\mathfrak{f}}(s,\psi')$ by deleting the Euler factors at all non-archimedean places in the support of $\mathfrak{m}$ that are not in the support of $\mathfrak{f}$. The most extreme case is to take ψ to be the trivial character $\mathbf{1}_{\mathfrak{m}}$ on $\mathrm{Cl}_{\mathfrak{m}}(K)$ with $\mathfrak{m} \ne 1$. In this case $\mathfrak{f}_\psi = 1$ and $L_{W,1}(s,\mathbf{1}_1) = \zeta_K(s)$, whereas $L_{W,\mathfrak{m}}(s,\mathbf{1}_{\mathfrak{m}})$ is missing the Euler factors at all non-archimedean places in $\mathrm{supp}(\mathfrak{m})$.

In the number field case, such ψ are valued in the unit circle since they have finite order (as $\mathrm{Cl}_{\mathfrak{m}}(K)$ is finite). In the function field case $\mathrm{Cl}_{\mathfrak{m}}(K)$ is a commutative extension of $q^{\mathbf{Z}}$ by a finite group, where the map $N : \mathrm{Cl}_{\mathfrak{m}}(K) \to q^{\mathbf{Z}}$ carries $[\mathfrak{p}_v]$ to q_v^{-1} and so ψ may not be valued in the unit circle. However, it is easy to see that in the function field case we may always write $\psi = \psi_0 N^{s_0}$ for some $s_0 \in \mathbf{C}$ and some ψ_0 with finite order.

Theorem 2.2. *The Euler product defining a Weber L-function $L_{W,\mathfrak{m}}(s,\psi)$ is absolutely and uniformly convergent in any half-plane* $\mathrm{Re}(s) \ge 1+\varepsilon$ *with $\varepsilon > 0$, provided that in the function field case ψ is valued in the unit circle. In the general function field case, if $\psi = \psi_0 N^{s_0}$ with ψ_0 of finite order then we must take* $\mathrm{Re}(s) \ge 1 - \mathrm{Re}(s_0) + \varepsilon$.

In particular, Weber L-functions are analytic and non-vanishing in their natural open half-planes of convergence.

Proof. The proof for ψ valued in the unit circle is identical to the classical case of Dirichlet characters, except that instead of expanding the product and comparing with the Riemann zeta-function $\zeta_{\mathbf{Q}}(s)$ after taking absolute values (using that ψ is valued in the unit circle) we now compare against $\zeta_K(s)$. The general function field case goes the same way. ■

We now turn to the more interesting case of abelian Artin L-functions. This will require some more care. Fix a separable closure K_s/K and let $G_K = \mathrm{Gal}(K_s/K)$. Consider a continuous character $\chi : G_K \to \mathbf{C}^\times$. This necessarily factors through a finite Galois group $\mathrm{Gal}(K'/K)$ by HW7 Exercise 4(i). We could view χ as a character of such a finite group $\mathrm{Gal}(K'/K)$, and sometimes this is convenient, but it is more canonical and more elegant (and more convenient for many purposes!) to view χ on G_K. For any non-archimedean place v, a choice of v' on K_s lifting v gives rise to subgroups $I(v'|v) \subseteq D(v'|v) \subseteq G_K$ that depend on v up to conjugation, and so in particular whether or not $\chi|_{I(v'|v)}$ is trivial is *independent* of the choice of v'. We say that χ is *unramified* at v when such triviality holds for some (and hence all) choice of v' over v; otherwise we say that χ is *ramified* at v. Since $\ker\chi$ is an open subgroup of G_K, we see that the non-archimedean v at which χ is ramified are those at which the finite extension K'/K corresponding to $\ker\chi$ is ramified; this is a *finite* set.

For each non-archimedean v at which χ is unramified, the restriction $\chi|_{D(v'|v)}$ is trivial on $I(v'|v)$ and so $\chi(\mathrm{Frob}(v'|v)) \in \mathbf{C}^\times$ makes sense (using any Frobenius element in $D(v'|v)$). If we change v' then $D(v'|v)$ gets conjugated in G_K and so $\mathrm{Frob}(v'|v)$ gets conjugated as well. But $\chi : G_K \to \mathbf{C}^\times$ is *conjugation-invariant* since $\mathbf{C}^\times$ is abelian, so $\chi(\mathrm{Frob}(v'|v)) \in \mathbf{C}^\times$ depends only on v; we shall write $\chi(\mathrm{Frob}_v)$ to denote this element for v at which χ is unramified.

Definition 2.3. For $\chi : G_K \to \mathbf{C}^\times$ a continuous character, let S_χ denote the union of the archimedean places and the finitely many ramified non-archimedean places for χ. The (abelian) *Artin L-function* associated to χ is

$$L(s,\chi) = \prod_{v \notin S_\chi} (1 - \chi(\mathrm{Frob}_v) q_v^{-s})^{-1}.$$

In the special case $\chi = 1$ there are no ramified places for χ and clearly $L(s,\chi) = \zeta_K(s)$.

Theorem 2.4. *The product defining abelian Artin L-functions is absolutely and uniformly convergent in each half-plane* $\mathrm{Re}(s) \geq 1+\varepsilon$ *with* $\varepsilon > 0$. *In particular, these functions are analytic and non-vanishing in* $\mathrm{Re}(s) > 1$.

Proof. The proof goes exactly as in the case of Weber L-functions, using comparison against $\zeta_K(s)$ after expanding the geometric series and taking absolute values everywhere. ∎

The nonsense with primitive versus imprimitive characters which plagued the theory of Weber L-functions (generalizing that disease for Dirichlet L-functions) is completely gone in Artin's setup. The two notions of L-function are related via class field theory, as will be seen later. Under this class field theory correspondence, each Artin L-function is the Weber L-function for a *primitive* class group character. This is seen very concretely in the special case $K = \mathbf{Q}$, where Weber's theory is essentially that of Dirichlet, as follows. In this case the Kronecker-Weber theorem identifies $G_{\mathbf{Q}}^{\mathrm{ab}}$ with $G_{\mathbf{Q}^{\mathrm{cyc}}/\mathbf{Q}}$ and so any Artin character $\chi : G_{\mathbf{Q}} \to \mathbf{C}^\times$ factors through a Dirichlet character $\psi : (\mathbf{Z}/m\mathbf{Z})^\times = \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}) \to \mathbf{Q}$ for some $m \geq 1$. There is a unique least such $m = f_\chi$ which divides all others, and this is essentially $\mathfrak{f}_\psi$ except that it ignores the archimedean information of whether the finite extension $\mathbf{Q}_\chi/\mathbf{Q}$ corresponding to $\ker\chi$ is totally real or not. Consider the unique ψ_χ compatible with χ at modulus $m = f_\chi$, so this is primitive in the sense of Dirichlet's viewpoint but possibly not quite in Weber's sense (where the modulus may contain ∞): if $\mathbf{Q}_\chi$ is totally real then ψ actually factors through $(\mathbf{Z}/f_\chi\mathbf{Z})^\times/\langle -1\rangle = \mathrm{Gal}(\mathbf{Q}(\zeta_{f_\chi})^+/\mathbf{Q})$. In other words, the classical dichotomy between even and odd (primitive) Dirichlet characters is actually measuring the finer sense of primitivity in the sense of Weber's theory.

Since the isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}) \simeq (\mathbf{Z}/m\mathbf{Z})^\times$ carries Frob_p to $p \bmod m$ for a positive prime $p \nmid m$, we have $\chi(\mathrm{Frob}_p) = \psi_\chi(p \bmod m)$. Hence, the compatibility with L-functions in this case works out to be:

$$L_{W,f_\chi}(s,\psi_\chi) = L(s,\chi).$$

Beware that the information at the real place is lurking in the background for the Weber L-function! It is hiding in the rule for specifying the correct Γ-factor to use to get a good functional equation. (The rule is this: if $\psi_\chi(-1) = 1$, which is to say if $\mathbf{Q}_\chi$ is totally real, then use $\pi^{-s/2}\Gamma(s/2)$ just as in the special case of $\zeta_{\mathbf{Q}}$, and if $\psi_\chi(-1) = -1$ then use $(2\pi)^{-s}\Gamma(s)$.)

3. Definition of Dirichlet density

Let K be a global field and let Σ be a set of non-archimedean places of K.

Definition 3.1. The *Dirichlet density* of Σ is

$$\delta_{\mathrm{Dir}}(\Sigma) = \lim_{s\to 1^+} \frac{\sum_{v\in\Sigma} q_v^{-s}}{\sum_v q_v^{-s}}$$

if this limit exists, where v ranges over non-archimedean places of K and q_v is the size of the residue field at v.

Of course, it should be explained why $\sum_v q_v^{-s}$ is finite for real $s > 1$. This sum is clearly bounded above by the product $\prod_{v\nmid\infty}(1-q_v^{-s})^{-1} = \zeta_K(s)$ (as one sees by formally expanding out the geometric series expansions for the factors), and in §2 it was shown how to prove the uniform and absolute convergence for $\zeta_K(s)$ with $\mathrm{Re}(s) > 1$ by giving upper bounds by powers of $\zeta_{\mathbf{Q}}(s)$ or $\zeta_{\mathbf{F}_p(t)}(s)$, which in turn can be directly estimated "by hand". In this way we see that the fraction in the definition of Dirichlet density makes sense for $\mathrm{Re}(s) > 1$. The denominator in the definition of Dirichlet density can be replaced by a more concrete quantity, exactly as in the classical case $K = \mathbf{Q}$:

Lemma 3.2. *As $s \to 1^+$, $\sum_v q_v^{-s} \sim \log(1/(s-1))$.*

Proof. By Hecke or Tate (and Weil for global function fields), ζ_K has a meromorphic continuation to $\mathbf{C}$ with a simple pole at $s = 1$. Hence, for real $s > 1$ we have $\log\zeta_K(s) \sim \log(1/(s-1))$ as $s \to 1^+$. Thus, it suffices to prove

$$\sum_v q_v^{-s} \overset{?}{\sim} \log\zeta_K(s) = \sum_v -\log(1-q_v^{-s})$$

as $s \to 1^+$. Since $-\log(1-q_v^{-s}) = q_v^{-s} + \sum_{j\geq 2} 1/(jq_v^{sj})$ for $s > 1$ and we know $\log\zeta_K(s) \sim \log(1/(s-1))$ explodes as $s \to 1^+$, it suffices to show that the sum $\sum_{j\geq 2}\sum_v q_v^{-sj}$ is bounded for s near 1. In fact, we shall show that it is absolutely and uniformly convergent for $\mathrm{Re}(s) \geq 1/2+\varepsilon$ for any $\varepsilon > 0$.

By expressing K as a finite separable extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$, and using that in a degree-d separable extension of global fields there are at most d places on the top field over a given place of the bottom field, with residue field degrees bounded by d as well, it suffices to treat the cases $K = \mathbf{Q}$ and $K = \mathbf{F}_p(t)$. Hence, it suffices to show that for real $s > 1/2$,

(1) the sum $\sum_{p,j\geq 2} p^{-js}$ is finite (this handles $K = \mathbf{Q}$),
(2) the sum $\sum_{r\geq 1, j\geq 2} p^r/(p^r)^{js}$ is finite for any prime p (this handles $K = \mathbf{F}_p(t)$, using the crude upper bound p^r on the number of places v of $\mathbf{F}_p(t)$ with $q_v = p^r$ for $r > 1$; there are $p+1$ places with $q_v = p$, due to the infinite place.)

For the first of these two sums, we have

$$\sum_{p,j\geq 2} p^{-js} = \sum_p \sum_{j\geq 2} p^{-js} \leq \sum_p \frac{2}{p^{js}} \leq 2\zeta_{\mathbf{Q}}(2s),$$

giving the desired finiteness for $s > 1/2$. For the second of the sums of interest, we take $s = 1/2+\varepsilon$ and get

$$\sum_{j\geq 2, r\geq 1} \frac{p^r}{(p^r)^{js}} = \sum_{j\geq 2, r\geq 1} \frac{1}{p^{rj(s-1/j)}} \leq \sum_{j\geq 2, r\geq 1} \frac{1}{p^{rj\varepsilon}} = \sum_{j\geq 2}\left(\frac{1}{1-p^{-\varepsilon j}} - 1\right).$$

Letting $a = p^{-\varepsilon} \in (0,1)$, we can rewrite this final sum as $\sum_{j\geq 2}((1-a^j)^{-1}-1)$, and so it suffices to prove finiteness of this latter sum for any $0 < a < 1$. Since $(1-a^j)^{-1} - 1 = a^j/(1-a^j)$ with $1/(1-a^j)$ bounded above for $j \geq 2$, we get an upper bound by a constant multiple of $\sum_{j\geq 2} a^j$, and this is obviously finite. ∎

We conclude that an equivalent definition of Dirichlet density is

$$\delta_{\mathrm{Dir}}(\Sigma) = \lim_{s\to 1^+} \frac{\sum_{v\in\Sigma} q_v^{-s}}{\log(1/(s-1))}$$

when this limit exists. Since $\log(1/(s-1)) \to \infty$ as $s \to 1^+$, changing Σ by a finite set does not impact whether or not it has a Dirichlet density, nor the value when this density exists. Hence, for statements concerning Dirichlet density it is typical to be sloppy concerning ambiguities with finite sets of places (such as ramified places in a finite separable extension of K).

Obviously finite sets have Dirichlet density zero, and in general Dirichlet density lies in the interval $[0, 1]$ when it exists. In particular, if a set Σ can be proved to have positive Dirichlet density, then it must be infinite. It is clear from the definition that a set Σ has a Dirichlet density if and only if its complement (in the set of non-archimedean places) has such a density, in which case the densities sum to 1. A subset of a set with Dirichlet density zero clearly has Dirichlet density that is moreover equal to 0, and a superset of a set with Dirichlet density 1 clearly has a Dirichlet density that is moreover equal to 1. It is easy to check that if $\Sigma_1, \dots, \Sigma_n$ are sets of non-archimedean places that admit Dirichlet densities $\delta_1, \dots, \delta_n$ and have overlaps $\Sigma_i \cap \Sigma_j$ with Dirichlet density 0 for $i \neq j$, then $\cup \Sigma_i$ has a Dirichlet density that is moreover equal to $\sum \delta_i$. However, Dirichlet density does not behave like a finitely additive measure. For example, if Σ and Σ' admit Dirichlet densities, it need not be the case that $\Sigma \cap \Sigma'$ or $\Sigma \cup \Sigma$ admit such densities (though clearly if one of them does then so does the other and the usual inclusion-exlcusion formula holds:

$$\delta(\Sigma_1 \cup \Sigma_2) = \delta(\Sigma_1) + \delta(\Sigma_2) - \delta(\Sigma_1 \cap \Sigma_2).)$$

Here is an interesting example of a set with full Dirichlet density:

Example 3.3. Let K'/K be a finite separable extension of global fields. The set Σ' of non-archimedean places v' on K' unramified over K and satisfying $f(v'|v) = 1$ has Dirichlet density 1 (where v is the place beneath v'); in the case that K'/K is Galois, these are precisely the places v' lying over the places v that are totally split in K'. Before we prove this, we warn that this density theorem for places of K' does *not* imply that the set of v in K that are totally split in K' has Dirichlet density 1. Indeed, as we shall see below, the Chebotarev density theorem will imply that if K'/K is Galois then the set of v in K that are totally split in K' has Dirichlet density $1/[K' : K]$. Thus, one should be careful to not confuse Dirichlet densities of sets on K with the sets over them on K'. Hence, it can be dangerous to view "Dirichlet density 1" as synonomous with "almost all places" when one moves between different global base fields.

To prove the density claim, we express K as a finite separable extension of a global field K_0 equal to $\mathbf{Q}$ or $\mathbf{F}_p(t)$, so K' is thereby realized as a finite separable extension of K_0. If v' on K' over v on K and over v_0 on K_0 satisfies $f(v'|v_0) = 1$ then obviously $f(v'|v) = 1$. Hence, it suffices to replace K with K_0, so we may assume $K = \mathbf{Q}$ or $K = \mathbf{F}_p(t)$. It is equivalent to show that the set Σ' of v' on K' with $f(v'|v) > 1$ has Dirichlet density 0. For such v' we have $q_{v'}^{-s} = q_v^{-f(v'|v)s} \leq q_v^{-2s}$ and there are at most $d = [K' : K]$ such v' over each v on K. Hence, the numerator $\sum_{v' \in \Sigma'} q_{v'}^{-s}$ in the definition of $\delta_{\mathrm{Dir}}(\Sigma')$ is bounded above by $[K' : K] \sum_v q_v^{-2s}$, and this is bounded for s near 1. Hence, dividing by $\log(1/(s-1))$ and sending $s \to 1^+$ gives a limit of 0.

There is an analogue of natural density for any global field K, as follows. For any $N > 0$, the set of non-archimedean places v on K that satisfy $q_v \leq N$ is a finite set. Indeed, by expressing K as a finite separable extension of $\mathbf{Q}$ or $\mathbf{F}_p(t)$ it suffices to treat these latter two cases, both of which are obvious. This finiteness result for each N permits us to define the *natural density* of a set Σ of non-archimedean places of K to be

$$\delta_{\mathrm{nat}}(\Sigma) = \lim_{x \to \infty} \frac{\#\{v \in \Sigma \mid q_v \leq N\}}{\#\{v \mid q_v \leq N\}}$$

if this limit exists. Beware, however, that natural density is a problematic notion for global function fields, even for $\mathbf{F}_q(t)$; e.g., Chebotarev's Density Theorem doesn't work for natural density. This issue is addressed is the paper "Competing prime asymptotic densities in $\mathbf{F}_q[x]$: a discussion" by C. Ballot in L'Enseignement Mathematique **54** (2008), pp. 303–328, where a "better" notion is suggested and discussed.

4. Artin L-functions

In the number field case, equidistribution questions for prime ideals in generalized ideal class groups are identified with such questions for Frobenius elements in finite abelian Galois groups by means of class field theory. Thus, we can focus our attention on this latter point of view, and so we prefer to think in terms of "abelian" Artin L-functions rather than Weber L-functions (although the two notions are basically the same). The key merit to the Galois side is that it admits an even further generalization of tremendous importance, namely equidistribution in *non-abelian* Galois groups. That is, if L/K is a finite Galois extension then we can consider Frobenius elements in $\mathrm{Gal}(L/K)$, or rather their conjugacy classes (since such elements are only well-defined up to conjugacy when labelled by places of K), and we can wonder how to formulate a suitable equidistribution result for these in terms of Dirichlet density. The Chebotarev density theorem (to be stated in §5) is this result, and its availability for non-abelian Galois extensions is very useful through number theory and arithmetic geometry (where it admits a vast generalization to the étale fundamental group of any connected scheme of finite type over $\mathbf{Z}$). It turns out, by a trick, that the proof of Chebotarev's theorem can be reduced to the case of abelian (and even cyclic) extensions. Nonetheless, now seems as good a time as any to present Artin's important generalization of his "abelian" L-functions $L(s,\chi)$ for characters $\chi : G_K \to \mathbf{C}^\times$ to the case of general Artin L-functions $L(s,\rho)$ associated to arbitrary continuous finite-dimensional complex representations $\rho : G_K \to \mathrm{GL}(V)$ of G_K. Such ρ are called *Artin representations*.

It is a general fact that $\mathrm{GL}_n(\mathbf{C})$ has "no small subgroups", which is to say that a neighborhood of 1 contains no nontrivial subgroups. In particular, just as in the case $n = 1$, any Artin representation ρ has open kernel and so is secretly just a representation of some finite Galois group $\mathrm{Gal}(L/K)$ in disguise. You will lose nothing if you simply take this as the definition of an Artin representation: simply require the kernel to be open and ignore the topology of $\mathbf{C}$ altogether. One consequence of factoring through a finite Galois group is that, exactly as in the case of dimension 1 that we considered earlier, $\rho|_{D(v'|v)}$ is trivial on $I(v'|v)$ for all but finitely many v. More specifically, the notions of ρ being *unramified* or *ramified* at a non-archimedean place v of K are defined (and justified to make sense) exactly as we did earlier in the 1-dimensional case, and ρ is ramified at a non-archimedean v if and only if the finite Galois extension K_ρ/K corresponding to $\ker\rho$ is ramified at v. Thus, ρ is unramified at all but finitely many places.

If ρ is unramified at a non-archimedean place v of K then we again get the element $\rho(\mathrm{Frob}(v'|v)) \in \mathrm{GL}(V)$ but now it is merely well-defined up to conjugacy since $\mathrm{GL}(V)$ is no longer abelian when $\dim V > 1$. Such conjugation ambiguity is eliminated by determinants, and in particular for v unramified in ρ we see that

$$\det(1 - \rho(\mathrm{Frob}(v'|v))q_v^{-s})^{-1}$$

makes perfectly good sense and is independent of the choice of v' over v on K_s. Artin made the crucial discovery that the "right" definition of $L(s,\rho)$ is generally *not* to simply form the product of such inverse-determinants at the unramifiied places (say at least when ρ is irreducible), but rather that (in contrast with the 1-dimensional case) there may be natural non-trivial factors to include at ramified places (even when ρ is irreducible). His observation is that although $\rho(\mathrm{Frob}(v'|v))$ is not well-defined up to conjugacy as an operator on V when $I(v'|v)$ acts nontrivially under ρ (i.e., v is ramified in ρ), it *is* well-defined up to conjugation on the subspace $V^{I(v'|v)}$. Indeed, since $I(v'|v)$ is normal in $D(v'|v)$ we see that the action on V by any choice of Frobenius element $\mathrm{Frob}(v'|v) \in D(v'|v)$ does preserve $V^{I(v'|v)}$, and that if we change the choice of Frobenius lift then the effect on $V^{I(v'|v)}$ is invisible since we can only change the choice of lift by scaling it against an element of $I(v'|v)$ (which does nothing on $V^{I(v'|v)}$!). Thus, for *all* non-archimedean places v we get a well-defined operator $\rho(\mathrm{Frob}(v'|v))$ on $V^{I(v'|v)}$, and if we change v' then everything conjugates compatibly. In particular,

$$\det(1 - \rho(\mathrm{Frob}(v'|v))q_v^{-s}|_{V^{I(v'|v)}})^{-1}$$

depends only on v and not on the choice v'. Note also that this inverse determinant is analytic for $\mathrm{Re}(s) > 0$ since $\rho(\mathrm{Frob}(v'|v))$ has eigenvalues that are roots of unity (as it is of finite order).

Definition 4.1. The *Artin L-function* of an Artin representation $\rho : G_K \to \mathrm{GL}(V)$ is

$$L(s,\rho) = \prod_{v\nmid\infty} \det(1 - \rho(\mathrm{Frob}(v'|v))q_v^{-s}|_{V^{I(v'|v)}})^{-1}.$$

It is understood that the local factor at v is 1 in case $V^{I(v'|v)} = 0$.

In the special case $\dim V = 1$ we may identify ρ with a continuous character $\chi : G_K \to \mathbf{C}^\times$, and then the above definition recovers $L(s,\chi)$. Indeed, the local Euler factors at the unramified primes are clearly the same, and since $\dim V = 1$ we have $V^{I(v'|v)} = 0$ in the ramified case.

Theorem 4.2. *The Euler product defining a general Artin L-function is absolutely and uniformly convergent in* $\mathrm{Re}(s) \geq 1 + \varepsilon$ *for any* $\varepsilon > 0$*. In particular, it is analytic and non-vanishing for* $\mathrm{Re}(s) > 1$.

Proof. For each v, $\rho(\mathrm{Frob}(v'|v))$ acting on $V^{I(v'|v)}$ has a finite set of eigenvalues that are all roots of unity. Thus, even though they cannot generally be chosen in a homomorphic manner (i.e., ρ may not be a direct sum of 1-dimensional representations), the old argument of expanding out an Euler product of geometric series and taking absolute values still works perfectly well. Instead of comparing against $\zeta_K(s)$ as in the 1-dimensional case, we now wind up comparing against $\zeta_K(s)^{\dim \rho}$. The desired conclusion then follows, as usual. ■

It is trivial to check that $L(s,\rho_1 \oplus \rho_2) = L(s,\rho_1)L(s,\rho_2)$; we say "$L$-functions are additive" (in the sense of carrying direct sums to products). Much less evident, but of great importance, if the behavior under induction. We merely state the result, leaving it to the literature (e.g., Milne's lecture notes) for the calculations to give a proof:

Theorem 4.3. *If K'/K is a finite extension inside of K_s, $\rho' : G_{K'} \to \mathrm{GL}(W)$ is an Artin representation, and $\mathrm{Ind}_{K'}^{K}(\rho') : G_K \to \mathrm{GL}(V)$ is the induced representation then*

$$L(s, \mathrm{Ind}_{K'}^{K}(\rho')) = L(s,\rho').$$

Remark 4.4. If v' is a place of K' that is ramified over its restriction v to K then even if ρ' is unramified at v' we will have that $\mathrm{Ind}_{K'}^{K}(\rho')$ is *ramified* at v. Hence, the validity of induction-invariance for Artin L-functions relies *crucially* on Artin's convention for defining possibly non-trivial Euler factors even at ramified places for the representation.

Example 4.5. Let ρ' be the trivial representation and assume K'/K is Galois. Then $\mathrm{Ind}_{K'}^{K}(\rho')$ is the regular representation of $\mathrm{Gal}(K'/K)$, which is $\oplus \rho^{\oplus \dim \rho}$ as ρ ranges over all irreducible representations of the finite group $\mathrm{Gal}(K'/K)$. Separating out the trivial representation from the rest, we get the useful identity

$$\zeta_{K'}(s) = \zeta_K(s) \prod_{\rho \neq 1} L(s,\rho)^{\dim \rho}$$

in which ρ ranges over the nontrivial irreducible representations of $\mathrm{Gal}(K'/K)$.

For example, if K'/K is an abelian extension then this says

$$\zeta_{K'}(s) = \zeta_K(s) \prod_{\chi \neq 1} L(s,\chi) \tag{1}$$

as χ ranges over all nontrivial characters $\chi : \mathrm{Gal}(K'/K) \to \mathbf{C}^\times$.

At the time of Artin's work, Hecke's thesis had already shown (in the number field case) that Weber L-functions for nontrivial characters are *holomorphic* on the entire complex plane (no poles), and that $\zeta_K(s)$ is meromorphic except for a simple pole at $s = 1$. By class field theory, the same holds for Artin L-functions of nontrivial 1-dimensional characters (as these are identified with Weber L-functions by means of class field theory); alternatively, one can replace Hecke's work with Tate's thesis to eliminate the reference to Weber L-functions, but one still crucially needs class field theory (to instead relate Artin's L-functions of Galois characters with the L-functions of adelic Hecke characters that were studied in Tate's thesis). Coupled with the identity (1), a comparison of orders at $s = 1$ then gives the striking result:

Corollary 4.6. *If $\chi_0 : G_K \to \mathbf{C}^\times$ is a nontrivial 1-dimensional continuous character then $L(1,\chi_0) \neq 0$.*

Proof. Let K'/K be the finite abelian extension corresponding to $\ker \chi_0$. Consider the identity (1). In this case the left side has a simple pole at $s = 1$, and the right side has the factor $\zeta_K(s)$ with a simple pole at $s = 1$ whereas all other factors (including $L(s,\chi_0)$ for $\chi = \chi_0$) are holomorphic at $s = 1$ and so *cannot vanish* at $s = 1$! ■

Inspired by this corollary, which is a vast generalization of Dirichlet's classical result of non-vanishing at $s = 1$ for L-functions of nontrivial Dirichlet characters (primitive or not, since the missing Euler factors are non-vanishing near $s = 1$), Artin was led to pose the following major unsolved problem:

Artin's conjecture: *If $\rho : G_K \to \mathrm{GL}(V)$ is an irreducible nontrivial Artin representation then $L(s, \rho)$ has a holomorphic continuation to the entire complex plane.*

The holomorphicity remains unsolved in general beyond the 1-dimensional case, except that in the function field case it was proved by Weil (and later reproved by Grothendieck using étale cohomology). Nowadays Artin's conjecture is viewed as a tiny piece of general conjectures of Langlands. The real power of Theorem 4.3 is realized when it is coupled with the additivity of Artin L-functions and the following result of Brauer which was conjectured by Artin for reasons we are going to explain shortly.

Theorem 4.7 (Brauer's induction theorem)**.** *Let G be a finite group and let χ be the character of a finite-dimensional complex representation of G. There exist* 1*-dimensional characters $\chi_i : H_i \to \mathbf{C}^\times$ on subgroups H_i of G and integers n_i such that*

$$\chi = \sum n_i \mathrm{Ind}_{H_i}^G(\chi_i)$$

as functions on G.

There are examples where the n_i's cannot all be taken to be non-negative. Nonetheless, this has fantastic applications. Namely, if $\rho : G_K \to \mathrm{GL}(V)$ is an Artin representation then by factoring it through some finite quotient $\mathrm{Gal}(K'/K)$ and applying Brauer's theorem to that finite group representation we get a finite collection of 1-dimensional characters $\chi_i : G_{K_i} \to \mathbf{C}^\times$ for various finite extensions K_i/K inside of K_s such that the character $\chi_\rho : G_K \to \mathbf{C}$ satisfies

$$\chi_\rho = \sum n_i \mathrm{Ind}_{K_i}^K(\chi_i)$$

as functions on G for some $n_i \in \mathbf{Z}$. Hence, by additivity of Artin L-functions and the induction-invariance of Artin L-functions (Theorem 4.3) we deduce

$$L(s, \rho) = \prod L(s, \chi_i)^{n_i}. \tag{2}$$

This is really an equality of infinite products indexed by the non-archimedean places of K, where for each $L(s, \chi_i)$ we collect all factors for places of K_i over a common place of K. But as we noted above, in the 1-dimensional case it was proved by Hecke (in the number field case, reproved and generalized by Tate) and Weil (in the function field case, reproved and generalized by Grothendieck) that $L(s, \chi)$ admits a meromorphic continuation to $\mathbf{C}$ (even holomorphic when $\chi \neq 1$), so (2) proves:

Corollary 4.8. *Artin L-functions have meromorphic continuation to $\mathbf{C}$.*

Since the exponents n_i in (2) cannot always be taken to be positive, we cannot do better than the meromorphicity by this essentially group-theoretic method (resting on Brauer's induction theorem). Using the precise form of the functional equation in the 1-dimensional case, one can use (2) to show that the meromorphic $L(s, \rho)$ has a functional equation with a precise form as predicted by Artin. But the holomorphicity when ρ is irreducible and nontrivial remains completely out of reach. Even for $K = \mathbf{Q}$ and $\dim \rho = 2$ it is not fully solved; the case when $\det \rho$ is an odd Dirichlet character was completely settled only very recently as a consequence of the solution to Serre's modularity conjecture, which in turn required the full force of work of Langlands and the deep improvements made in recent years in Wiles' modularity methods. And the case when $\det \rho$ is even is hopeless by these methods!

5. Chebotarev's density theorem

For each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ and a positive prime $p \nmid N$, we have $\sigma = \mathrm{Frob}_p$ if and only if the isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \simeq (\mathbf{Z}/N\mathbf{Z})^\times$ carries σ to $p \bmod N$. Hence, an equivalent statement of Dirichlet's theorem on primes in arithmetic progressions is this:

Theorem 5.1 (Dirichlet)**.** *Choose $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ and let Σ be the set of prime ideals $v \nmid N$ of $\mathbf{Z}$ such that σ is the Frobenius element for v in $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$. The set Σ has a Dirichlet density, and it is equal to $1/[\mathbf{Q}(\zeta_N) : \mathbf{Q}]$.*

By using the formalism of Artin L-functions and class field theory, together with Corollary 4.6, one can prove a vast generalization:

Theorem 5.2 (Chebotarev)**.** *Let K'/K be a finite Galois extension of global fields. Let c be a conjugacy class in $G = \mathrm{Gal}(K'/K)$. Let Σ_c be the set of non-archimedean places v of K that are unramified in K' and have Frobenius conjugacy class c. The set Σ_c has a Dirichlet density and it is equal to $|c|/|G|$. In particular, Σ_c is infinite.*

The proof of this theorem involves a clever reduction to the case when K'/K is abelian, and even cyclic, at which point class field theory gets used.

Corollary 5.3. *Let K'/K be a finite Galois extension of global fields. The set of v in K that are totally split in K' has Dirichlet density $1/[K' : K]$. In particular, it is infinite.*

Proof. This is the special case $c = \{1\}$. ∎

We can generalize this corollary slightly, dropping the Galois condition:

Corollary 5.4. *Let K'/K be a finite separable extension of global fields. The set of v in K that are totally split in K' has a positive Dirichlet density equal to $1/[K'' : K]$ with K''/K a Galois closure of K'/K, and in particular it is an infinite set.*

Proof. Let K''/K' be a Galois closure of K'/K, so v is unramified in K'' if and only if it is unramified in K'. For such v, the condition that v be totally split in K' is that it be totally split in K'' (since K'' is a compositum of extensions of K that are abstractly isomorphic to K'). Hence, this set of v's has Dirichlet density $1/[K'' : K]$. ∎

Let K be a global field and let S be a finite set of places that contains the archimedean places. Let $G_{K,S} = \mathrm{Gal}(K_S/K)$ be the Galois group for a maximal extension K_S/K unramified outside S; here, $K_S \subseteq K_{\mathrm{sep}}$ is the compositum of finite subextensions K'/K that are unramified outside S. For each $v \notin S$ and v' on K_S extending v, we get a well-defined Frobenius element $\phi(v'|v)$ in $G_{K,S}$. As we vary v', these sweep out a conjugacy class, called the *Frobenius conjugacy class* for v in $G_{K,S}$.

Theorem 5.5. *The set of Frobenius elements $\phi(v'|v) \in G_{K,S}$ is dense with respect to the Krull topology.*

In the language of modern algebraic geometry, $G_{K,S}$ is the étale fundamental group of the "punctured curve" Spec $\mathscr{O}_{K,S}$ (with respect to the geometric generic point Spec K_{sep} as base point). Hence, this theorem is an analogue of the obvious fact that the topological fundamental group of a finitely-punctured Riemann surface is generated by loops (which play the role of Frobenius elements).

Proof. By the definition of the Krull topology, it has to be proved that for each finite Galois subextension K'/K, every element of $\mathrm{Gal}(K'/K)$ is the image of $\phi(v'|v)$ for some v' on K_S over some $v \notin S$. By the functoriality of Frobenius with respect to passage to the quotient, for any v' on K_S over $v \notin S$, say with restriction w on K', the image of $\phi(v'|v)$ in $\mathrm{Gal}(K'/K)$ is $\phi(w|v)$. Hence, it is equivalent to prove that every element of $\mathrm{Gal}(K'/K)$ is a Frobenius element for a place over some $v \notin S$. The Chebotarev density theorem shows even more: every element of $\mathrm{Gal}(K'/K)$ is a Frobenius element relative to infinitely many places of K unramified in K'. ∎

Example 5.6. Let $K/\mathbf{Q}$ be a finite Galois extension and let q be a prime of $\mathbf{Q}$ unramified in K. Choose an integer a not divisible by q, and choose $g \in \mathrm{Gal}(K/\mathbf{Q})$. We claim that there exist infinitely many positive primes p unramified in K such that $p \equiv a \bmod q$ and $g = \phi(\mathfrak{p}|p\mathbf{Z})$ for a prime $\mathfrak{p}$ of $\mathscr{O}_K$ over p.

The point is that since $\mathbf{Q}(\zeta_q)$ is totally ramified at q, it is linearly disjoint from K over $\mathbf{Q}$. Hence, the natural map $\mathbf{Q}(\zeta_q) \otimes_{\mathbf{Q}} K \to K(\zeta_q)$ is an isomorphism, and more specifically the natural map

$$\mathrm{Gal}(K(\zeta_q)/\mathbf{Q}) \to \mathrm{Gal}(K/\mathbf{Q}) \times \mathrm{Gal}(\mathbf{Q}(\zeta_q)/\mathbf{Q})$$

is an isomorphism. We may consider the ordered pair $(g, a \bmod q)$ on the right side as corresponding to an element γ in the Galois group on the left side, and the desired properties for p say exactly that it is unramified in $K(\zeta_q)$ and there exists a prime over p in $K(\zeta_q)$ whose Frobenius element in $\mathrm{Gal}(K(\zeta_q)/\mathbf{Q})$ is γ. Thus, applying Chebotarev's theorem to the Galois extension $K(\zeta_q)/\mathbf{Q}$ does the job.