Math 676. A compactness theorem for the idele group

1. Introduction

Let K be a global field, so $K^\times$ is naturally a discrete subgroup of the idele group $\mathbf{A}_K^\times$ and by the product formula it lies in the kernel $(\mathbf{A}_K^\times)^1$ of the continuous idelic norm

$$\|\cdot\|_K : \mathbf{A}_K^\times \to \mathbf{R}_{>0}^\times.$$

We saw in class that if S is a finite non-empty set of places of K that contains all archimedean places, then the *combined* statement that $\mathscr{O}_{K,S}$ has finite class group and finitely generated unit group with rank $|S|-1$ is logically equivalent to the assertion (that does not involve S) that the quotient $(\mathbf{A}_K^\times)^1/K^\times$ is *compact.* (Some aspects of that proof are addressed in §5 below.)

In the case of number fields we saw how to directly prove that $\mathscr{O}_{K,S}$ has finite class group and $\mathscr{O}_{K,S}^\times$ is finitely generated with rank $|S|-1$ for any S containing the archimedean places. These methods rest on Minkowski's lemma, and so to carry them over to global function fields one needs a generalization of Minkowski's lemma. Our aim in this handout is to bypass the problem by giving a purely adelic proof of:

Theorem 1.1. *For any global field K, $(\mathbf{A}_K^\times)^1/K^\times$ is compact.*

The proof will be uniform across all global fields, and the key to the proof is an adelic replacement for the role of Minkowski's lemma in the classical argument for number fields. In particular, this gives a new proof of Dirichlet's unit theorem and finiteness of class groups for rings of S-integers $\mathscr{O}_{K,S}$. However, if one strips away the adelic language in the case of number fields (especially when S is precisely the set of archimedean places) then one essentially recovers the classical argument. It must be emphasized that the power of the adelic approach is that it is more systematic across all global fields K and it is methodologically *simpler.* Also, this approach shows the close logical connection between the two fundamental finiteness theorems of algebraic number theory, a closeness that one cannot fully appreciate until one has come across their relation with Theorem 1.1. Our exposition of the proof of Theorem 1.1 follows Chapter 2 (§14–§17) in the book *Algebraic number theory* edited by Cassels and Frohlich.

2. The adelic Minkowski lemma

The classical Minkowski lemma concerns compact quotients V/Λ with V a finite-dimensional $\mathbf{R}$-vector space and Λ a lattice in V: if μ_Λ is the Haar measure on V that is adapted to counting measure on Λ and the volume-1 measure on V/Λ, then for $X \subseteq V$ is compact, convex, and symmetric about the origin with $\mu_\Lambda(X) > 2^{\dim V}$ the intersection $X \cap \Lambda$ is nonzero.

As a special case, let $V = K \otimes_{\mathbf{Q}} \mathbf{R} \simeq \prod_{v|\infty} K_v$ for a number field K and let $\Lambda = \mathscr{O}_K$. For any $C > 0$ and any $\xi = (\xi_v) \in \prod_{v|\infty} K_v$ with $\prod_v \|\xi_v\|_v > C$ (where $\|\cdot\|_v$ is the square of the standard absolute value for complex v) we may consider the compact, convex, centrally symmetric region $X_\xi \subseteq V$ consisting of points (x_v) such that $\|x_v\|_v \le \|\xi_v\|_v$ for all $v|\infty$. The volume of X_ξ may be universally bounded below in terms of C and arithmetic invariants of K, so by taking C sufficiently large we can ensure that X_ξ has large enough volume to satisfy the requirements for Minkowski's lemma. Hence, for such large C depending only on K we know that X_ξ contains a nonzero element of $\Lambda = \mathscr{O}_K$, which is to say that there exists $x \in \mathscr{O}_K - \{0\}$ with $\|x\|_v \le \|\xi_v\|_v$ for all $v|\infty$. The adelic result we shall now prove is simply a variant on this final assertion, with the set of archimedean factors replaced by the adele ring and the discrete co-compact subgroup $\mathscr{O}_K \subseteq K \otimes_{\mathbf{Q}} \mathbf{R}$ replaced with the discrete co-compact subgroup $K \subseteq \mathbf{A}_K$.

Theorem 2.1 (Minkowski). *Let K be a global field. For $\xi = (\xi_v) \in \mathbf{A}_K^\times$, define the closed subset*

$$X_\xi = \{(x_v) \in \mathbf{A}_K \mid \|x_v\|_v \le \|\xi_v\|_v\} \subseteq \mathbf{A}_K.$$

There exists $C = C_K > 0$ such that if $\|\xi\|_K > C$ then $X_\xi \cap K$ contains a nonzero element.

Since $\|\xi_v\|_v = 1$ for all but finitely many v, it is clear that X_ξ is a product of closed discs that coincide with $\mathscr{O}_v$ for all but finitely many non-archimedean v, so X_ξ is compact. However, observe that the hypotheses on the idele ξ only concerns $\|\xi\|_K$, and hence the set of non-archimedean v's for which $\xi_v \in \mathscr{O}_v^\times$ is not controlled by the assumptions on ξ.

Proof. Let μ be the unique Haar measure on $\mathbf{A}_K$ that is adapted to counting measure on the discrete subgroup K and the volume-1 measure on the compact quotient $\mathbf{A}_K/K$. Let $Z \subseteq \mathbf{A}_K$ denote the compact set of adeles $z = (z_v)$ such that $\|z_v\|_v \leq 1$ for non-archimedean v, $\|z_v\|_v \leq \|1/2\|_v$ for $v|\infty$, so if $z, z' \in Z$ then $\|z_v - z'_v\|_v \leq 1$ for all v. Since Z is compact and contains an open neighborhood around the origin, $\mu(Z)$ is finite and positive. Moreover, this volume is intrinsic to K.

By Lemma 2.2 below, $\mu(\xi Z) = \|\xi\|_K \mu(Z)$. Thus, by taking $C = 1/\mu(Z)$ we have $\mu(\xi Z) > 1$. We claim that this forces the existence of a pair of distinct elements in ξZ with the same image in $\mathbf{A}_K/K$, which is to say that the projection map $\pi : \xi Z \to \mathbf{A}_K/K$ has some fiber with size at least 2. Indeed, if χ on $\mathbf{A}_K$ is the characteristic function of the subset ξZ then Fubini's theorem gives

$$\mu(\xi Z) = \int_{\mathbf{A}_K} \chi \mathrm{d}\mu = \int_{\mathbf{A}_K/K} (\sum_{c \in K} \chi(c + x)) \mathrm{d}\overline{\mu}(\overline{x}) = \int_{\mathbf{A}_K/K} \#\pi^{-1}(\overline{x}) \mathrm{d}\overline{\mu}(\overline{x})$$

with $\overline{\mu}$ the volume-1 Haar measure on $\mathbf{A}_K/K$, and so if all fibers of π have size at most 1 then we get $\mu(\xi Z) \leq \int_{\mathbf{A}_K/K} \mathrm{d}\overline{\mu} = 1$, contradicting that $\mu(\xi Z) > 1$.

We conclude that there exists $x, x' \in \xi Z$ such that $x - x' = a \in K^\times$. Thus, if we write $x = \xi z$ and $x' = \xi z'$ with $z, z' \in Z$ then

$$\|a\|_v = \|\xi_v(z_v - z'_v)\|_v = \|\xi\|_v \|z_v - z'_v\|_v \leq \|\xi\|_v$$

for all places v. Hence, $a \in X_\xi \cap K^\times$. ■

The following pleasing lemma was used in the preceding proof.

Lemma 2.2. *For $\xi \in \mathbf{A}_K^\times$, the scaling effect of ξ on Haar measures of $\mathbf{A}_K$ is $\|\xi\|_K$.*

Proof. From the local theory we know that the scaling effect of $\xi_v \in K_v^\times$ on Haar measures of K_v is $\|\xi_v\|_v$. Hence, we want to build a Haar measure on $\mathbf{A}_K$ as a "product measure" of local measures on the K_v's so as to compute the scaling action on $\mathbf{A}_K$ as the product of scalings along the local "factors" of $\mathbf{A}_K$. Strictly speaking $\mathbf{A}_K$ is not a product space, so we need to be a little careful.

Choose a Haar measure μ_v on K_v for each place v such that for all but finitely many non-archimedean place v the compact open subring $\mathscr{O}_v \subseteq K_v$ is assigned volume 1. Hence, for each finite set of places S on K containing the archimedean places it makes sense to form the product Borel measure μ_S on the open subring $\mathbf{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathscr{O}_v$ by using μ_v on K_v for $v \in S$ and μ_v on $\mathscr{O}_v$ for $v \notin S$ (as all but finitely many of these $\mathscr{O}_v$'s are assigned volume 1). Clearly each μ_S is a Borel measure on $\mathbf{A}_{K,S}$, and if S' contains S then $\mu_{S'}$ on $\mathbf{A}_{K,S'}$ restricts to μ_S on the open subset $\mathbf{A}_{K,S}$. Since the $\mathbf{A}_{K,S}$'s are a directed system of open additive subgroups that exhaust $\mathbf{A}_K$, by using the correspondence between Borel measures and positive functionals on the space of continuous functions with compact support (or by working explicitly) it is straightforward to check that there exists a unique Borel measure μ on $\mathbf{A}_K$ that restricts to μ_S on $\mathbf{A}_{K,S}$ for all S. Clearly μ is positive on non-empty opens, is finite on compacts, and is translation-invariant because each μ_S has these properties on $\mathbf{A}_{K,S}$ for each S. Since $\mathbf{A}_K$ (and each $\mathbf{A}_{K,S}$) has a countable base of opens, it follows that μ is σ-regular on $\mathbf{A}_K$ (as is μ_S on $\mathbf{A}_{K,S}$ for each S). Hence, μ is a Haar measure on $\mathbf{A}_K$ (and μ_S is a Haar measure on $\mathbf{A}_{K,S}$ for each S).

Now the problem is a simple computation. We consider S so large so that $\xi_v \in \mathscr{O}_v^\times$ for all $v \notin S$. Since $\xi \in \mathbf{A}_{K,S}^\times$, with $\mathbf{A}_{K,S}$ an open subgroup of $\mathbf{A}_K$, the scaling effect of ξ on μ coincides with the scaling effect of ξ on μ_S. Since μ_S is a genuine product measure, we may choose compact neighborhood $N = \prod_v N_v$ of the origin in $\mathbf{A}_{K,S}$ that is a product of compact neighborhoods N_v of the origin in each factor of $\mathbf{A}_{K,S}$ with local factor $N_v = \mathscr{O}_v$ for all $v \notin S$. Thus, clearly

$$\mu_S(\xi N) = \prod_v \mu_v(\xi_v N_v) = \prod_v \|\xi_v\|_v \mu_v(N_v) = \|\xi\|_K \mu_S(N)$$

with $\mu_S(N)$ finite and positive. Hence, the desired scaling factor is indeed $\|\xi\|_K$. ■

Observe that the proof of Lemma 2.2 does not use the product formula. Hence, we can use it to give a pretty measure-theoretic proof of the product formula as follows. Let μ be the unique Haar measure on $\mathbf{A}_K$ compatible with counting measure on the discrete subgroup K and with volume-1 measure on the compact quotient $\mathbf{A}_K/K$. For any $a \in K^\times$, multiplication by a on $\mathbf{A}_K$ carries K to itself isomorphically induces an automorphism of $\mathbf{A}_K/K$, so by Fubini's theorem we conclude that the scaling effect of a on μ is the product of its scaling effects on counting measure for K and the volume-1 measure for $\mathbf{A}_K/K$. These latter two measures are obviously invariant under arbitrary topological group automorphisms, and hence the scaling effect of a on μ must be 1. Thus, by Lemma 2.2 we recover the product formula: $\|a\|_K = 1$!

Before we turn to the proof of Theorem 1.1, it seems worthwhile to record an auxiliary consequence of the adelic Minkowski lemma that helps us to appreciate the delicate nature of the discreteness of K in $\mathbf{A}_K$: it is crucial for such discreteness that we included *all* places of K, in the sense that omitting a single factor has a dramatic consequence:

Theorem 2.3. *If v_0 is a place of K and $\mathbf{A}_K^{v_0}$ is the factor ring of $\mathbf{A}_K$ obtained by deleting the factor K_{v_0} then the diagonal embedding of K into $\mathbf{A}_K^{v_0}$ has dense image.*

This result is called the *strong approximation theorem* for the adele ring because it is much stronger than the usual weak approximation theorem: it says that not only can we find an element $x \in K$ that is as close as we please to choices of elements $x_v \in K_v$ for any large finite set S of places $v \neq v_0$, but we can simultaneously ensure that x is v-integral at all remaining non-archimedean places $v \neq v_0$. Since we have no use for strong approximation here (though it is a prototype for very important approximation results for adelic groups later in life), we refer the reader to §15 in Chapter 2 of Cassels–Frohlich for details on its proof.

3. Proof of Theorem 1.1

We are now in position to prove the compactness of $(\mathbf{A}_K^\times)^1/K^\times$ for any global field K. The crucial ingredient is:

Lemma 3.1. *The kernel $(\mathbf{A}_K^\times)^1$ of the idelic norm inherits the* same *topology regardless of whether we view it in $\mathbf{A}_K^\times$ (where it is a closed subgroup) or in $\mathbf{A}_K$, and it is closed in $\mathbf{A}_K$.*

This is an interesting property since $\mathbf{A}_K^\times$ with its topological group structure certainly does not have the subspace topology from $\mathbf{A}_K$ (with respect to which inversion is not continuous).

Proof. Let us first check closedness of the subset $(\mathbf{A}_K^\times)^1$ in the topological space $\mathbf{A}_K$. We choose $x = (x_v) \in \mathbf{A}_K$ not in $(\mathbf{A}_K^\times)^1$, and we seek a neighborhood N of x in $\mathbf{A}_K$ disjoint from $(\mathbf{A}_K^\times)^1$. Let S be a finite set of places containing all archimedean places and such that $x_v \in \mathscr{O}_v$ for all $v \notin S$.

First assume $x_{v_0} = 0$ for some v_0. Consider $N = \prod N_v$ where $N_v = \mathscr{O}_v$ for all $v \notin S \cup \{v_0\}$, N_v is a compact neighborhood of x_v in K_v for $v \in S$ with $v \neq v_0$, and N_{v_0} is a very small neighborhood of the origin in K_{v_0}. Clearly any idele in N has idelic norm bounded above by the product of local norms along factors in S and along v_0, so by taking N_{v_0} to be very small we can ensure that any idele in N has idelic norm very near 0 and in particular not equal to 1.

Now assume $x_v \in K_v^\times$ for all v. Since $\|x_v\|_v \leq 1$ for all but finitely many v, the infinite product $\prod_v \|x_v\|_v$ has partial products that eventually form a monotonically decreasing sequence, and hence this product makes sense as a non-negative real number. If it is less than 1 then we take S to be a finite set of places containing all archimedean places and such that $x_v \in \mathscr{O}_v$ for all $v \notin S$ and $\prod_{v \in S} \|x_v\|_v < 1$. We now may argue as in the preceding paragraph, except we take $N_v = \mathscr{O}_v$ for all $v \notin S$ and we choose N_v to be a very small neighborhood of x_v for $v \in S$. If instead $\prod_v \|x_v\|_v = P > 1$ then convergence of this infinite product provides a large finite set of places S (containing all archimedean places) such that $x_v \in \mathscr{O}_v$ for all $v \notin S$, $1/2P > 1/q_v$ for all $v \notin S$ (so $\|\xi_v\|_v < 1$ forces $\|\xi_v\|_v < 1/2P$), and $\prod_{v \in S} \|x_v\|_v \in (1, 2P)$. Hence, by choosing $N_v = \mathscr{O}_v$ for $v \notin S$ and choosing N_v very small around x_v for all $v \in S$ we may ensure that if ξ is an idele in N and $\|\xi_v\|_v < 1$ for some $v \notin S$ then

$$\|\xi\|_K = \prod_{v \in S} \|\xi_v\|_v \cdot \prod_{v \notin S} \|\xi_v\|_v < 2P/2P = 1,$$

whereas if $\|\xi_v\|_v \geq 1$ for all $v \notin S$ then $\|\xi\|_K > 1$. In particular, every idele in N has idelic norm not equal to 1. This completes the proof that $(\mathbf{A}_K^\times)^1$ is a closed subset of $\mathbf{A}_K$.

Now we turn to the proof that the inclusion of $(\mathbf{A}_K^\times)^1 \hookrightarrow \mathbf{A}_K$ is a homeomorphism onto its (closed) image. This map is certainly continuous since the definition of the topology on $\mathbf{A}_K$ is as a subset of $\mathbf{A}_K \times \mathbf{A}_K$ and the projection onto the first factor is a continuous map. Hence, our problem is to prove that every neighborhood $N \subseteq (\mathbf{A}_K^\times)^1$ around a point x contains the intersection of $(\mathbf{A}_K^\times)^1$ with a neighborhood of x in $\mathbf{A}_K$. Multiplication by $1/x$ is an automorphism of $\mathbf{A}_K$ that carries $(\mathbf{A}_K^\times)^1$ homeomorphically onto itself with respect to the subspace topology from the topological group $\mathbf{A}_K^\times$, so without loss of generality we can assume $x = 1$.

Due to the description of a neighborhood-basis of the identity in $\mathbf{A}_K^\times$, we can shrink N so that it has the form $N = (\prod_v N_v) \cap (\mathbf{A}_K^\times)^1$ with N_v equal to a small disc centered at 1 in $K_v^\times$ for all v in a large finite set of places S that contains all archimedean places, and with $N_v = \mathscr{O}_v^\times$ for all $v \notin S$. By shrinking the N_v's we can ensure that $\prod_{v \in S} \|\xi_v\|_v < 2$ for all $\xi \in \prod_v N_v \subseteq \mathbf{A}_K^\times$, yet for $\xi \in \prod_v N_v$ clearly $\|\xi\|_K = \prod_{v\in S} \|\xi_v\|_v$ because $N_v = \mathscr{O}_v^\times$ for all $v \notin S$. Thus, if we let W be the neighborhood $\prod_{v\in S} N_v \times \prod_{v \notin S} \mathscr{O}_v$ of 1 in $\mathbf{A}_K$ then any $\xi \in W \cap (\mathbf{A}_K^\times)^1$ satisfies

$$1 = \|\xi\|_K = \prod_{v\in S} \|\xi_v\|_v \cdot \prod_{v\notin S} \|\xi_v\|_v < 2 \prod_{v\notin S} \|\xi_v\|_v.$$

Since $\|\xi_v\|_v \le 1$ for all $v \notin S$, we get $1 < 2\|\xi_{v_0}\|_{v_0}$ for all $v_0 \notin S$, so $1 \ge \|\xi_{v_0}\| > 1/2 \ge 1/q_{v_0}$ and hence $\xi_{v_0} \in \mathscr{O}_{v_0}^\times = N_{v_0}$ for all $v_0 \notin S$. In other words,

$$W \cap (\mathbf{A}_K^\times)^1 \subseteq (\prod_v N_v) \cap (\mathbf{A}_K^\times)^1 = N,$$

as desired. ■

With Lemma 3.1 proved, Theorem 1.1 may now be proved. We are aiming to prove that the quotient $(\mathbf{A}_K^\times)^1/K^\times$ is compact, and the topology on $(\mathbf{A}_K^\times)^1$ is initially given by that on $\mathbf{A}_K^\times$. By Lemma 3.1, this topology is also induced by $\mathbf{A}_K$ with $(\mathbf{A}_K^\times)^1$ closed in $\mathbf{A}_K$. Hence, for any compact subset $W \subseteq \mathbf{A}_K$ we have that $W \cap (\mathbf{A}_K^\times)^1$ is a compact set (as it is closed in W). It is therefore sufficient to find such a large W for which the projection map

$$W \cap (\mathbf{A}_K^\times)^1 \to (\mathbf{A}_K^\times)^1/K^\times$$

is surjective. Choose a constant $C > 0$ as in the adelic Minkowski lemma (Theorem 2.1), and choose any idele $\xi \in \mathbf{A}_K^\times$ such that $\|\xi\|_K > C$. Define the compact set

$$W = \{x = (x_v) \in \mathbf{A}_K \mid \|x_v\|_v \le \|\xi_v\|_v \text{ for all } v\}$$

in $\mathbf{A}_K$. For any $\theta = (\theta_v) \in (\mathbf{A}_K^\times)^1$ the idele $\theta^{-1}\xi$ has idelic norm $\|\xi\|_K > C$, so by the adelic Minkowski lemma there exists nonzero $a \in K$ such that $\|a\|_v \le \|\theta_v^{-1}\xi_v\|_v$ for all v, and hence $a\theta \in W$. Since $a \in K^\times \subseteq (\mathbf{A}_K^\times)^1$, it follows that $a\theta \in W \cap (\mathbf{A}_K^\times)^1$ is a representative of the class of θ in $(\mathbf{A}_K^\times)^1/K^\times$. This concludes the proof of Theorem 1.1.

4. Variation in K

In global class field theory, a fundamental object of interest is the idele class group $\mathbf{A}_K^\times/K^\times$. We wish to explain how Theorem 1.1 helps us to prove a pleasant feature of this topological group as we vary K. Fix a finite separable extension K'/K. Since $K' \otimes_K \mathbf{A}_K \to \mathbf{A}_{K'}$ is a (topological) isomorphism, it follows that $K' \cap \mathbf{A}_K = K$ inside of $\mathbf{A}_{K'}$. Hence, the natural continuous map of topological groups

$$\mathbf{A}_K^\times/K^\times \to \mathbf{A}_{K'}^\times/K'^\times$$

is injective. The topological isomorphism $K' \otimes_K \mathbf{A}_K \to \mathbf{A}_{K'}$ ensures that the natural continuous ring map $\mathbf{A}_K \to \mathbf{A}_{K'}$ is a closed embedding, and clearly $\mathbf{A}_K \cap \mathbf{A}_{K'}^\times = \mathbf{A}_K^\times$. Hence, by the definition of the idelic topology we conclude that the natural continuous group map $\mathbf{A}_K^\times \to \mathbf{A}_{K'}^\times$ is a closed embedding. It is not obvious "by hand" whether the induced continuous injection modulo the multiplicative groups of the global fields is again a closed embedding, and this point seems to not be addressed in any of the basic books on ideles, so we now prove that nothing unpleasant happens:

Theorem 4.1. *The natural map* $\mathbf{A}_K^\times/K^\times \to \mathbf{A}_{K'}^\times/K'^\times$ *is a closed embedding.*

Proof. The natural continuous group map $(\mathbf{A}_K^\times)^1 \to (\mathbf{A}_{K'}^\times)^1$ induces a continuous injection

$$(\mathbf{A}_K^\times)^1/K^\times \to (\mathbf{A}_{K'}^\times)^1/K'^\times$$

yet source and target are compact Hausdorff, so this map is a closed embedding. We shall now distinguish the cases of number fields and global function fields, due to the different nature of the image of the idelic norm in $\mathbf{R}_{>0}^\times$ in the two cases.

First consider the case of global function fields. The idelic norm has discrete image in $\mathbf{R}_{>0}^\times$, and hence $(\mathbf{A}_K^\times)^1/K^\times$ is open and closed in $\mathbf{A}_K^\times/K^\times$ with infinite cyclic cokernel. Since the map $\mathbf{A}_K^\times \to \mathbf{A}_{K'}^\times$ intertwines with the two idelic norms through the map of raising to the $[K' : K]$th-power (because $\mathrm{N}_{K'/K}(x') = x'^{[K':K]}$ for $x' \in K'$ and for the polynomial map $\mathrm{N}_{K'/K} : K' \to K$ between finite-dimensional K-vector spaces), it follows that under the map of interest in the theorem the preimage of $(\mathbf{A}_{K'}^\times)^1/K'^\times$ is $(\mathbf{A}_K^\times)^1/K^\times$, and similarly with other cosets for fixed values of the idelic norms. Hence, the property of being a closed embedding may be checked on the level of the open and closed compact kernels for the idelic norm, for which we have already seen the result to hold for compactness reasons.

Now consider the case of number fields. We fix an archimedean place v_0 on K and an archimedean place v_0' over it on K', and we use the continuous splitting of the idelic norm $\mathbf{A}_K^\times \to \mathbf{R}_{>0}^\times$ via the inclusion that sends $t \in \mathbf{R}_{>0}^\times$ to the idele that is 1 in factors away from v_0 and is $t^{1/[K_{v_0}:\mathbf{R}]}$ in $\mathbf{R}_{>0}^\times \subseteq K_{v_0}^\times$ in the factor at v_0. The resulting topological group isomorphisms

$$\mathbf{A}_K^\times/K^\times \simeq \mathbf{R}_{>0}^\times \times (\mathbf{A}_K^\times)^1/K^\times, \quad \mathbf{A}_{K'}^\times/K'^\times \simeq \mathbf{R}_{>0}^\times \times (\mathbf{A}_{K'}^\times)^1/K'^\times$$

carry the map of interest in the theorem over to the product of the maps given by the $[K' : K]$th-power map on $\mathbf{R}_{>0}^\times$ and the natural map on the kernel factors for the idelic norms. This map along the second factor has been seen to be a closed embedding due to compactness, and the map $x \mapsto x^{[K':K]}$ on $\mathbf{R}_{>0}^\times$ is trivially a closed embedding (even a topological isomorphism). Hence, the product map of interest is a closed embedding. ■

5. From ideles to S-units

We conclude by relating compactness of the idele class group to the finiteness theorems for class groups and unit groups of rings of S-integers of global fields. As above, we fix a global field K and let S be a finite non-empty set of places of K that contains the archimedean places. The discussion that follows does not rest on any of the results proved above.

Lemma 5.1. *The subring* $\mathscr{O}_{K,S} \subseteq \prod_{v\in S} K_v$ *is discrete and co-compact. Also, the subgroup* $\mathscr{O}_{K,S}^\times$ *of* S*-units is discrete in the closed subgroup* $(\prod_{v\in S} K_v^\times)^1$ *of elements* $(x_v)_{v\in S}$ *satisfying* $\prod_{v\in S} |x_v|_v = 1$.

Proof. The topology on $\prod_{v\in S} K_v^\times$ is the subspace topology in $\prod_{v\in S} K_v$, so if $\mathscr{O}_{K,S}$ inherits the discrete topology from $\prod_{v\in S} K_v$ then the locus $\mathscr{O}_{K,S} - \{0\}$ where it meets $\prod_{v\in S} K_v^\times$ inherits the discrete topology from $\prod_{v\in S} K_v^\times$. In particular, $\mathscr{O}_{K,S}^\times$ would inherit the discrete topology from $\prod_{v\in S} K_v^\times$ and so also would inherit the discrete topology from the closed subgroup $(\prod_{v\in S} K_v^\times)^1$. Thus, we now may focus our attention on the additive problem.

Since $\prod_{v\in S} K_v$ is Hausdorff, a subgroup is discrete if and only if it meets a neighborhood of 0 in a finite set. We know that K is discrete in $\mathbf{A}_K$ (as this was inferred from the cases $K = \mathbf{Q}$ and $K = \mathbf{F}_p(x)$ in the homework), so the open subring $\mathbf{A}_{K,S}$ of S-adeles (that meets K in $\mathscr{O}_{K,S}$) contains the ring of S-integers $\mathscr{O}_{K,S}$ as a discrete subring. Hence, any compact neighborhood of the origin in $\mathbf{A}_{K,S}$ meets $\mathscr{O}_{K,S}$ in a finite set. Let $N_v \subseteq K_v$ be a compact neighborhood of 0 for each $v \in S$, so

$$N = \prod_{v\in S} N_v \times \prod_{v\notin S} \mathscr{O}_v$$

is a compact neighborhood of 0 in $\mathbf{A}_{K,S}$. Clearly $\prod_{v\in S} N_v$ is a neighborhood of 0 in $\prod_{v\in S} K_v$, and it meets $\mathscr{O}_{K,S}$ in exactly the set where $\mathscr{O}_{K,S}$ meets N inside of $\mathbf{A}_{K,S}$. Thus, $\prod_{v\in S} K_v$ has a neighborhood of 0 meeting $\mathscr{O}_{K,S}$ in a finite set, as desired. ∎

Let $\mathscr{L}_S : \prod_{v\in S} K_v^\times \to \mathbf{R}^S$ be the continuous logarithm map

$$\mathscr{L}_S : (x_v)_{v\in S} \mapsto \sum_{v\in S} \log|x_v|_v.$$

The preimage of the hyperplane $H_S = \{(t_v) \in \mathbf{R}^S \mid \sum_{v\in S} t_v = 0\}$ is precisely $(\prod_{v\in S} K_v^\times)^1$. The continuous map $\mathscr{L}_S$ between locally compact Hausdorff spaces is proper (in the sense that the preimage of a compact set is compact) because bounding $\log|x_v|_v$ above and below puts x_v inside of a compact annulus in $K_v^\times$. Hence, the induced continuous map of topological groups $\mathscr{L}'_S : (\prod_{v\in S} K_v^\times)^1 \to H_S$ is proper. Since $\mathscr{L}'_S$ is proper and $\mathscr{O}_{K,S}^\times$ is discrete in $(\prod_{v\in S} K_v^\times)^1$, it follows (as we saw earlier in the number field case) that $\mathscr{L}_S(\mathscr{O}_{K,S}^\times)$ is discrete in H_S.

The topology on H_S is that arising from its structure of finite-dimensional $\mathbf{R}$-vector space of dimension $|S|-1$, and so any discrete subgroup of H_S is finitely generated with dimension at most $|S|-1$. In particular, this shows that $\mathscr{L}_S(\mathscr{O}_{K,S}^\times)$ is finitely generated with rank at most $|S|-1$, and equality is attained if and only if $H_S/\mathscr{L}_S(\mathscr{O}_{K,S}^\times)$ is compact. Since $\mathscr{L}_S$ is proper, its kernel $\mathscr{L}_S^{-1}(0)$ is compact and hence meets the discrete subgroup $\mathscr{O}_{K,S}^\times$ in a finite set. Thus, $\mathscr{L}_S(\mathscr{O}_{K,S}^\times)$ is the quotient of $\mathscr{O}_{K,S}^\times$ by a finite subgroup, and thus we see that $\mathscr{O}_{K,S}^\times$ is finitely generated with rank at most $|S|-1$. In fact:

Lemma 5.2. *The rank of the finitely generated group $\mathscr{O}_{K,S}^\times$ is equal to $|S|-1$ if and only if $(\prod_{v\in S} K_v^\times)^1/\mathscr{O}_{K,S}^\times$ is compact.*

Proof. Consider the continuous map

$$\overline{\mathscr{L}}'_S : (\prod_{v\in S} K_v^\times)^1/\mathscr{O}_{K,S}^\times \to H_S/\mathscr{L}_S(\mathscr{O}_{K,S}^\times)$$

induced by $\mathscr{L}'_S$. We have to prove that the source is compact if and only if the target is compact. Since $\mathscr{L}_S(\mathscr{O}_{K,S}^\times) = \mathscr{L}'_S(\mathscr{O}_{K,S}^\times)$ is discrete in H_S and the map $\mathscr{L}'_S$ is a proper map between locally compact Hausdorff spaces, it is easy to check that $\overline{\mathscr{L}}'_S$ is proper. Thus, if the target of $\overline{\mathscr{L}}'_S$ is proper then so is its source.

Conversely, assume that $(\prod_{v\in S} K_v^\times)^1/\mathscr{O}_{K,S}^\times$ is compact. Its image in the locally compact Hausdorff group $H_S/\mathscr{O}_{K,S}^\times$ is therefore a compact closed (!) subgroup. Thus, $\mathscr{L}_S((\prod_{v\in S} K_v^\times)^1)$ is a closed subgroup of H_S, and moreover $H_S/\mathscr{O}_{K,S}^\times$ is compact if and only if the locally compact and Hausdorff quotient $H_S/\mathscr{L}_S((\prod_{v\in S} K_v^\times)^1)$ is compact. Since H_S is a finite-dimensional vector space over $\mathbf{R}$ with dimension $|S|-1$, to prove compactness of the Hausdorff quotient of H_S modulo the closed subgroup $\mathscr{L}_S((\prod_{v\in S} K_v^\times)^1)$ it suffices to exhibit a set of $|S|-1$ linearly independent vectors in this closed subgroup. The case $|S|=1$ is trivial (explicitly, $H_S = \{0\}$ in this case), so we may assume $|S|>1$.

Pick $v_0 \in S$ with $v_0|\infty$ in the number field case. For each $v \in S-\{v_0\}$ we will find a vector $h(v) \in H_S \subseteq \mathbf{R}^S$ lying in the image of $\mathscr{L}_S$ such that the only non-zero coordinates of $h(v)$ are in positions v and v_0; this provides the desired linearly independent set. To find $h(v)$ we merely have to show that the subgroups $|K_{v_0}^\times|_{v_0}$ and $|K_v^\times|_v$ are $\mathbf{Z}$-linearly dependent inside of $\mathbf{R}_{>0}$ (in the sense that there exists a non-trivial multiplicative dependence relation between non-trivial elements of these two groups). This is obvious in the number field case, as $|K_{v_0}^\times|_{v_0} = \mathbf{R}_{>0}$ due to the condition $v_0|\infty$. In the global function field case, all value groups $|K_v^\times|_v$ are finite-index subgroups of $q^{\mathbf{Z}}$ (with q denoting the size of the constant field in K), and hence any two are $\mathbf{Z}$-linearly dependent. ∎

By Lemma 5.1 and Lemma 5.2, we see that $\mathscr{O}_{K,S}^\times$ is a discrete and finitely generated subgroup of $(\prod_{v\in S} K_v^\times)^1$ with rank at most $|S|-1$, and that this rank bound is an equality if and only if the locally compact Hausdorff quotient $(\prod_{v\in S} K_v^\times)^1/\mathscr{O}_{K,S}^\times$ is compact. Since $K^\times$ is discrete in $\mathbf{A}_K^\times$, it follows that $\mathscr{O}_{K,S}^\times$ is discrete in the open subgroup $\mathbf{A}_{K,S}^\times$. There is an evident continuous quotient map from the locally compact *Hausdorff* group $(\mathbf{A}_{K,S}^{\times})^1/\mathscr{O}_{K,S}^{\times}$ onto $(\prod_{v\in S} K_v^{\times})^1/\mathscr{O}_{K,S}^{\times}$ with kernel that is compact (as the kernel is a continuous image of the compact group $\prod_{v\in S-S_\infty} \mathscr{O}_v^{\times}$), so compactness of $(\prod_{v\in S} K_v^{\times})^1/\mathscr{O}_{K,S}^{\times}$ is equivalent to compactness of $(\mathbf{A}_{K,S}^{\times})^1/\mathscr{O}_{K,S}^{\times}$.

Consideration of the exact sequence of locally compact Hausdorff topological groups

$$1 \to (\mathbf{A}_{K,S}^{\times})^1/\mathscr{O}_{K,S}^{\times} \to (\mathbf{A}_K^{\times})^1/K^{\times} \to (\mathbf{A}_K^{\times})^1/K^{\times}\cdot(\mathbf{A}_{K,S}^{\times})^1 \to 1$$

(with open inclusion on the left and discrete quotient on the right) shows that compactness of the left side is equivalent to the conjunction of compactness of the middle term and finiteness of the right side (as a discrete space is compact if and only if it is finite). Thus, we conclude that compactness of $(\mathbf{A}_K^{\times})^1/K^{\times}$ is logically equivalent to the finitely generated group $\mathscr{O}_{K,S}^{\times}$ having rank $|S|-1$ and the discrete group $(\mathbf{A}_K^{\times})^1/K^{\times}\cdot(\mathbf{A}_{K,S}^{\times})^1$ being finite. There is an evident isomorphism $\mathbf{A}_K^{\times}/K^{\times}\cdot\mathbf{A}_{K,S}^{\times} \simeq \mathrm{Pic}(\mathscr{O}_{K,S})$ induced by $(x_v) \mapsto \prod_{v\notin S} \mathfrak{p}_v^{\mathrm{ord}_v(x_v)}$, and the subgroup inclusion of discrete groups

$$(\mathbf{A}_K^{\times})^1/K^{\times}\cdot(\mathbf{A}_{K,S}^{\times})^1 \subseteq \mathbf{A}_K^{\times}/K^{\times}\cdot\mathbf{A}_{K,S}^{\times}$$

(why is this injective?) has cokernel $\mathbf{A}_K^{\times}/\mathbf{A}_{K,S}^{\times}\cdot(\mathbf{A}_K^{\times})^1 = \|\mathbf{A}_K^{\times}\|_K/\|\mathbf{A}_{K,S}^{\times}\|_K$ that is *finite* by inspection (since S is non-empty and contains S_∞). Hence, $\mathrm{Pic}(\mathscr{O}_{K,S})$ is finite if and only if the finite-index discrete subgroup $(\mathbf{A}_K^{\times})^1/K^{\times}\cdot(\mathbf{A}_{K,S}^{\times})^1$ is finite. Putting everything together, this shows:

Theorem 5.3. *For a global field K and a non-empty finite set of places S that contains the set of archimedean places of K, the compactness of $(\mathbf{A}_K^{\times})^1/K^{\times}$ is logically equivalent to the combined assertion that* $\mathrm{Pic}(\mathscr{O}_{K,S})$ *is finite and that the finitely generated group $\mathscr{O}_{K,S}^{\times}$ has rank $|S|-1$.*