Math 249A. Homework 8

1. Let A be an abelian variety over a field k, and G a closed k-subgroup scheme. We now construct A/G.

(i) Show that if $f : A' \to A$ is an isogeny and $G' = f^{-1}(G)$, then A'/G' exists if and only if A/G does, and then they are uniquely isomorphic respecting the quotient maps and f. (Hint: $A = A'/(\ker f)$).

(ii) Let $B = G^0_{\mathrm{red}}$, which we know (by Exercise 1(ii), HW5) is an abelian subvariety. By Poincaré reducibility, there is an isogeny-complement: an abelian subvariety B' in A over k such that $B \times B' \to A$ is an isogeny. Using (i), reduce to the case $A = B \times B'$ with $B \subseteq G$ and $G \cap B'$ finite.

(iii) In the special case at the end of (ii), prove $G = B \times (G \cap B')$, and deduce that A/G always exists.

2. Let k be a field.

(i) Let K/k be an extension, X is a k-scheme, and $Z \subseteq X_K$ a closed subscheme. Let Σ be the set of intermediate fields $K/F/k$ such that Z descends (necessarily uniquely) to a closed subscheme of X_F. Prove that the intersection of any collection of elements of Σ is again in Σ (note this formulation allows for working Zariski-locally on X!), and deduce that Σ contains a unique minimal element contained in all others; this is called the (minimal) *field of definition* of Z over k. Hint: reduce to a fact in linear algebra about "field of definition" for subspaces of a vector space.

(ii) Applying (i) to graphs of morphisms, deduce that if X and X' are k-schemes with X separated, then for any extension field K/k and K-morphism $f : X'_K \to X_K$, among all intermediate fields $K/F/k$ such that f descends (necessarily uniquely!) to an F-morphism $X_F \to X'_F$ there is one such F contained in all others. It is called the (minimal) *field of definition* of f over k.

(iii) Improve Exercise 2(iv) in HW5 by proving that if A and B are abelian varieties over k and $f : A_K \to B_K$ is a homomorphism over an extension K/k then f is defined over the separable closure of k in K. Deduce that if K/k is *primary* (i.e., k is separably algebraically closed in K) then the functor $A \rightsquigarrow A_K$ from abelian varieties over k to abelian varieties over K is *fully faithful*. Hence, it is functorial (!) to speak of an abelian variety over such a K being "defined" over k. Using that abelian subvarieties are images of endomorphisms (Poincaré reducibility!), deduce that every abelian subvariety of A_K is defined over k.

(iv) Let A be an abelian variety over k, and K/k a primary extension. Prove that for any isogeny $f : A_K \to \mathscr{B}$ over K with $\mathrm{char}(k) \nmid \deg f$ there is up to unique isomorphism a pair (B, i) consisting of an abelian variety B over k and a K-isomorphism $i : \mathscr{B} \simeq B_K$, and that $i \circ f$ is defined over k. (Hint: use étale torsion-levels.) Then use double-duality to prove a similar result for isogenies $\mathscr{B}' \to A_K$.

(v) Let E be a supersingular elliptic curve over a field k of characteristic $p > 0$. It is known that $\ker F_{E/k} = \alpha_p$. For any extension K/k, show that the set of nonzero proper K-subgroups $G \subsetneq (\alpha_p \times \alpha_p)_K$ is in bijection with the set of lines in K^2 (via $G \mapsto \mathrm{T}_e(G)$), and deduce that if K/k is primary and $K \neq k$ then $(E \times E)_K$ admits an isogenous quotient *not* defined over k as an abstract abelian variety over K.

3. Let A be an abelian variety of dimension $g > 0$ over a field k, and $F \subseteq \mathrm{End}^0_k(A)$ be a commutative $\mathbf{Q}$-subalgebra of the endomorphism algebra such that F is semisimple as a ring (i.e., a finite product of fields). For $\ell \neq \mathrm{char}(k)$, let $F_\ell := \mathbf{Q}_\ell \otimes_{\mathbf{Q}} F$; note that F_ℓ is semisimple too. Let $D = \mathrm{End}^0_k(A)$ be the endomorphism algebra, so D is a finite-dimensional semisimple $\mathbf{Q}$-algebra.

(i) Using the injectivity of $\mathbf{Q}_\ell \otimes_{\mathbf{Q}} D \to \mathrm{End}_{\mathbf{Q}_\ell}(\mathrm{V}_\ell(A))$, prove that $\mathrm{V}_\ell(A)$ is a faithful F_ℓ-module.

(ii) Prove that $[F : \mathbf{Q}] \leq 2g$, with equality if and only if $\mathrm{V}_\ell(A)$ is free of rank 1 as an F_ℓ-module for some ℓ, in which case the same holds for all ℓ. When equality holds, prove that F is a maximal commutative subalgebra of D. In such cases we say that A has *sufficiently many complex multiplications*.

(iii) Define an abelian variety B over k to be *isotypic* if its k-simple factors are pairwise k-isogenous. (This property can be destroyed by a finite ground field extension, unless k is finite.) Prove that the set $\{A_i\}$ of maximal isotypic abelian subvarieties over A over k is finite and that $\prod A_i \to A$ is an isogeny, with $D = \prod \mathrm{End}^0_k(A_i)$. In case $[F : \mathbf{Q}] = 2g$, show that $F = \prod F_i$ with F_i a commutative subfield in $\mathrm{End}^0_k(A_i)$ satisfying $[F_i : \mathbf{Q}] = 2\dim(A_i)$ (so F_i is maximal commutative in $\mathrm{End}^0_k(A_i)$). In particular, if F is a field then show that A is isotypic and remains so after any extension on k! (If A is isotypic and admits sufficiently many complex multiplications then structure theory for semisimple algebras shows that the k-simple factors of A admit sufficiently many complex multiplications. Much deeper is that F can be chosen to be a CM field for such A.)