Math 249A. Homework 6

1. A map of abelian varieties $f : A \to B$ over k is an *isogeny* if it is surjective with finite kernel on $\overline{k}$-points, or equivalently (as we have seen earlier in the course) if f is finite and flat.

(i) Using the theorem of Deligne from Exercise 5 in HW4 and the "quotient" property for fpqc homomorphisms between group schemes (as discussed in class), prove that if $\dim A = \dim B$ then f is an isogeny if and only there exists $g : B \to A$ such that $g \circ f = [n]_A$, in which case $f \circ g = [n]_B$.

(ii) Let ℓ be a prime with $\ell \neq \mathrm{char}(k)$. Prove that f is an isogeny if and only if the induced map $\mathrm{T}_\ell(f) : \mathrm{T}_\ell(A) \to \mathrm{T}_\ell(B)$ on ℓ-adic Tate modules is injective with finite cokernel, and equivalently if and only if $\mathrm{V}_\ell(f) : \mathrm{V}_\ell(A) \to \mathrm{V}_\ell(B)$ is an isomorphism. In such cases, prove that $\deg f$ is not divisible by ℓ if and only if $\mathrm{T}_\ell(f)$ is an isomorphism. (There are analogues for $\ell = \mathrm{char}(k) > 0$, using Dieudonné modules.)

(iii) The *isogeny category* of abelian varieties over k has objects the abelian varieties over k and morphisms $\mathrm{Hom}^0(A, B) := \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Hom}_k(A, B)$. Explain why this forms a category, prove that the "forgetful" functor from the category of abelian varieties over k to the isogeny category is faithful but not fully faithful, and that a map of abelian varieties is an isomorphism in the isogeny category if and only if it is an isogeny.

2. Let S be a scheme, X an S-scheme, and G an S-group scheme. Assume there is given a left action map $G \times_S X \to X$. This action is called *free* if $G(T)$ acts freely on $X(T)$ for all S-schemes T.

(i) Prove that freeness is equivalent to the map $G \times_S X \to X \times_S X$ defined by $(\gamma, x) \mapsto (\gamma.x, x)$ being a monomorphism, and deduce that freeness is insensitive to fpqc base change. (Hint: in a category with fiber products, a map is a monomorphism if and only if its relative diagonal is an isomorphism.)

(ii) Let X be a scheme locally of finite type over an algebraically closed k, equipped with an action by a k-group G locally of finite type. For each $x \in X(k)$, prove that the functor assigning to any k-scheme S the subgroup of $g \in G(S)$ fixing $x_S \in X(S)$ is represented by a closed k-subgroup G_x, the *isotropy group scheme* at x. Explain why G_x naturally acts on the tangent space $\mathrm{T}_x(X)$ (viewed as an affine space over k), so the action map $G_x \to \mathrm{GL}(\mathrm{T}_x(X))$ defines a map of Lie algebras $\mathrm{Lie}(G_x) \to \mathfrak{gl}(\mathrm{T}_x(X))$ (i.e., an action in the sense of Lie algebra representations of $\mathrm{Lie}(G_x)$ on $\mathrm{T}_x(X)$).

Prove that the action is free if and only if $G(k)$ acts freely on $X(k)$ and $\mathrm{Lie}(G_x)$ acts freely on $\mathrm{T}_x(X)$ for all $x \in X(k)$ (i.e., nonzero elements of $\mathrm{Lie}(G_x)$ act without nonzero fixed points on $\mathrm{T}_x(X)$).

(iii) Assume $G \to S$ is fpqc and G acts freely on X. A *quotient* of X by the G-action is a G-invariant fpqc map $\pi : X \to \overline{X}$ such that $G \times_S X \to X \times_{\overline{X}} X$ defined by $(g, x) \mapsto (g.x, x)$ is an isomorphism. Prove that such a quotient, if it exists, is unique up to unique isomorphism, is initial among G-invariant maps from X to S-schemes, and retains the quotient property after base change to any S-scheme.

3. Let A be an abelian variety over k, and G a *finite* k-subgroup scheme of A. This exercise proves the existence and uniqueness of a quotient abelian variety A/G, and considers an important example.

(i) Prove that up to unique isomorphism there is at most one pair $(\overline{A}, \pi)$ consisting of an abelian variety $\overline{A}$ and a surjective k-homomorphism $\pi : A \to \overline{A}$ with $G = \ker \pi$. Prove that if it exists then it is necessarily a quotient in the strong sense of Exercise 2(iii). Conversely, prove that if there is a quotient A/G in the strong sense of Exercise 2(iii) then it is necessarily an abelian variety. (Hint: a noetherian ring is regular if it admits a faithfully flat regular extension, by Theorem 23.7 of Matsumura CRT, and a k-algebra is finite type if it admits a faithfully flat extension of finite type over k, by Prop. 9.1 in Exposé V of SGA3.)

(ii) Choose $n \in \mathbf{Z} - \{0\}$ killing G (e.g., the order of G), and consider the quotient mapping $[n]_A : A \to A$ that identifies A with $A/A[n]$ (in particular, $A/A[n]$ exists and is an abelian variety). Explain how this identifies the problem of existence of A/G in the sense of (i) with the quotient problem from Exercise 2(iii) for the action of the A-group $G \times A$ on A viewed as an A-scheme via $[n]_A : A \to A$. The existence of quotients of free actions by finite flat group schemes on schemes affine (even finite!) over a noetherian base is solved in general by Theorem 4.1 in Exposé V of SGA3 (you can read §1–§4 there without the earlier exposés.)

(iii) Let $\mathscr{L}$ be an ample line bundle on A, so $K(\mathscr{L})$ is a finite subgroup scheme of A. Deduce that the dual abelian variety $\widehat{A}$ is naturally identified with the quotient $A/K(\mathscr{L})$. (In Mumford's book, he develops from scratch a good theory of quotients of abelian varieties modulo finite subgroup schemes and then proves directly for ample $\mathscr{L}$ that the quotient $A/K(\mathscr{L})$ satisfies the required properties to be a dual abelian variety. In this way he constructs the theory of the dual abelian variety without using the theory of Picard schemes.)