Math 249A. Homework 3

1. Let X be a scheme over a field k, and assume that $X(k)$ is dense in X (e.g., $k = k_s$ with X geometrically reduced and locally of finite type). Prove that $X(k)$ is "relatively schematically dense" in X in the following sense: for any k-scheme S, if a closed subscheme Z of X_S contains all sections in $X_S(S) = X(S)$ arising from $X(k)$ then $Z = X_S$.

2. Let $f : X \to S$ be a proper flat map of schemes, with S locally noetherian, and assume that the geometric fibers of f are reduced and connected. Assume projectivity and/or noetherian hypotheses if you wish (but not needed).

(i) Using the theory of cohomology and base change, prove that $\mathscr{O}_S = f_*\mathscr{O}_X$ via the natural map.

(ii) Let $\mathscr{L}$ be a line bundle on X. Prove that $\mathscr{L} \simeq f^*(\mathscr{N})$ for a line bundle $\mathscr{N}$ on S if and only if $f_*(\mathscr{L})$ is invertible and the natural map $f^*f_*(\mathscr{L}) \to \mathscr{L}$ is an isomorphism (in which case $\mathscr{N} \simeq f_*\mathscr{L}$). Deduce that in such cases, the formation of $f_*(\mathscr{L})$ commutes with any base change on S.

3. Let Y be a normal locally noetherian separated scheme, and U a dense affine open in Y. Prove that $Y - U$ has pure codimension 1 in the sense that its generic points have codimension 1 in Y (i.e., local ring of Y at such points is 1-dimensional).

4. Let $X \to S$ be a map of schemes and $Z \subseteq Z'$ a containment of S-flat closed subschemes whose associated ideal sheaves are locally finitely generated. If $Z_s = Z'_s$ inside of X_s for all $s \in S$ then prove that $Z = Z'$ inside of X. (Hint: rename Z' as X, and use Nakayama's Lemma.)

5. Let X be a smooth proper and geometrically connected curve of genus g over a field k such that $X(k) \neq \emptyset$, and let $P = \mathrm{Pic}_{X/k}$ be its Picard scheme. We have seen in earlier homework that the k-group scheme P is smooth of dimension $\dim \mathrm{H}^1(X, \mathscr{O}_X) = g$ over k and that P^0 is proper, so P^0 is an abelian variety of dimension g. In this exercise we identify $P^0(k)$ as a subgroup of $P(k) = \mathrm{Pic}(X)$.

(i) For any k-scheme S and section $x \in X(S) = X_S(S)$, prove that the quasi-coherent ideal sheaf of the closed subscheme $x : S \hookrightarrow X_S$ is an invertible sheaf whose local generators are nowhere zero divisors on $\mathscr{O}_{X_S}$. (Hint: use flatness of $x(S)$ over S to show that the formation of the ideal sheaf commutes with any base change on S, and then use Nakayama's Lemma to show the sheaf is locally principal. For invertibility, use the local flatness criterion in the form of Theorem 22.5 in Matsumura's CRT after reducing to the case of noetherian S.) The inverse of this ideal sheaf is denoted $\mathscr{O}_{X_S}(x)$.

(ii) For a coherent sheaf $\mathscr{F}$ on a proper k-scheme Y, recall that the *Euler characteristic* $\chi(\mathscr{F})$ is defined to be $\sum(-1)^j h^j(Y, \mathscr{F})$. For an invertible sheaf $\mathscr{L}$ on X, prove that $\chi(\mathscr{L}^n) = d_{\mathscr{L}} \cdot n + (1 - g)$ for an integer $d_{\mathscr{L}}$; we call this integer the *degree* of $\mathscr{L}$. Likewise, for a Weil divisor $D = \sum n_i x_i$ on our curve X, define $\deg(D) = \sum n_i [k(x_i) : k]$. Prove that both notions of degree are invariant under extension of the ground field, and that they coincide when $Y = X$ and $\mathscr{L} \simeq \mathscr{O}_X(D)$.

(iii) Choose $e \in X(k)$, and define $X^g \to P$ by defining $X(S)^g \to P(S) = \mathrm{Pic}(X_S)/\mathrm{Pic}(S)$ for any k-scheme S to be

$$(x_1, \dots, x_g) \mapsto \mathscr{O}_{X_S}(x_1) \otimes \cdots \otimes \mathscr{O}_{X_S}(x_g) \otimes \mathscr{O}_{X_S}(e)^{\otimes(-g)}.$$

This map carries $(e, \dots, e)$ to $0 \in P^0(k)$, so by connectedness of X^g this map factors through a map $X^g \to P^0$ between proper k-schemes. Using the Riemann-Roch theorem for $X_{\overline{k}}$, prove that this latter map on $\overline{k}$-points hits exactly the line bundles on $X_{\overline{k}}$ of degree 0; don't ignore the case $g = 0$.

(iv) It is a general fact (proved in Ch. II, §5) that the Euler characteristic is locally constant for a flat coherent sheaf relative to a proper morphism of locally noetherian schemes. Deduce that there is a well-defined map of k-group schemes from P to the constant group $\mathbf{Z}$ over $\operatorname{Spec} k$ assigning to any point of $P(S)$ the locally constant function given by the fiberwise degree of the line bundle. Using that $\mathbf{Z}$ as a k-scheme contains no nontrivial k-proper subgroups, prove that for any field K, $P^0(K)$ is the subgroup of degree-0 line bundles in $\mathrm{Pic}(X_K)$. (This depends crucially on the hypothesis that $X(k) \neq \emptyset$; Grothendieck gave a way to define $P = \mathrm{Pic}_{X/k}$ without such a hypothesis on X, and then $P^0(k)$ can fail to have this concrete description when $\mathrm{Br}(k) \neq 1$.)