Math 249A. A height bound

Let X be a projective scheme over a global field K, and let $\mathscr{L}$ be a line bundle on X. The *base locus* $B \subseteq X$ of $\mathscr{L}$ is the support of the coherent sheaf

$$\mathrm{coker}(\mathscr{O}_X \otimes_k \mathrm{H}^0(X, \mathscr{L}) \to \mathscr{L}) = \{x \in X \mid s(x) = 0 \text{ in } \mathscr{L}(x) \text{ for all } s \in \mathrm{H}^0(X, \mathscr{L})\}.$$

In other words, for the natural map $f : X - B \to \mathbf{P}(\mathrm{H}^0(X, \mathscr{L}))$ (given by $x \mapsto [s_0(x), \dots, s_n(x)]$ for a K-basis $\{s_0, \dots, s_n\}$ of $\mathrm{H}^0(X, \mathscr{L})$) and the canonical isomorphism $\theta : f^*(\mathscr{O}(1)) \simeq \mathscr{L}$, there is no strictly large open set in X across which θ extends. In this handout, we aim to prove that $h_{K,\mathscr{L}}$ is bounded below on $(X-B)(\overline{K})$. Since K will be fixed throughout the discussion, we write $h_{\mathscr{L}}$ instead of $h_{K,\mathscr{L}}$. (It is natural to try to relate $h_{\mathscr{L}}|_{(X-B)(\overline{K})}$ to the composition of $X - B \to \mathbf{P}(\mathrm{H}^0(X, \mathscr{L}))$ and a standard height on the latter projective space, but we do not address that here.)

Pick a pair of very ample line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on X such that $\mathscr{L} \simeq \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$, so $h_{\mathscr{L}} = h_{\mathscr{L}_1} - h_{\mathscr{L}_2}$ (as functions $X(\overline{K}) \to \mathbf{R}$ modulo bounded functions). Thus, we need to show for some specific representatives $H_{\mathscr{L}_i}$ of the equivalence class $h_{\mathscr{L}_i}$ that $H_{\mathscr{L}_1} \geq H_{\mathscr{L}_2} + c$ on $(X - B)(\overline{K})$, for some $c \in \mathbf{R}$. If $\{U_1, \dots, U_n\}$ is an open cover of $X - B$ then it suffices to do this for each $U_i(\overline{K})$ separately, allowing the constant c and the choice of representatives $H_{\mathscr{L}_i}$ to depend on i. Letting s vary through a basis of $\mathrm{H}^0(X, \mathscr{L})$, the corresponding open sets $X_s = \{x \in X \mid s(x) \neq 0\}$ cover $X - B$. Hence, it suffices to work on X_s for a fixed choice of nonzero $s \in \mathrm{H}^0(X, \mathscr{L})$.

Multiplication by s defines an injection of sheaves $\mathscr{L}_2 \to \mathscr{L}_1$ and hence a K-linear injection $\mathrm{H}^0(X, \mathscr{L}_2) \to \mathrm{H}^0(X, \mathscr{L}_1)$. Let $\{\sigma_0, \dots, \sigma_n\}$ be a K-basis of $\mathrm{H}^0(X, \mathscr{L}_2)$, so $\{s\sigma_i\}$ is a K-linearly independent set in $\mathrm{H}^0(X, \mathscr{L}_2)$ and thus extends to a K-basis

$$\{s\sigma_0, \dots, s\sigma_n, \tau_1, \dots, \tau_m\}$$

of $\mathrm{H}^0(X, \mathscr{L}_2)$. We use these specific ordered bases to define $H_{\mathscr{L}_1}$ and $H_{\mathscr{L}_2}$ as the restrictions to $X(\overline{K})$ of the *standard height functions* via the canonical embeddings

$$X \hookrightarrow \mathbf{P}(\mathrm{H}^0(X, \mathscr{L}_1) \simeq \mathbf{P}^n_K, \quad X \hookrightarrow \mathbf{P}(\mathrm{H}^0(X, \mathscr{L}_2) \simeq \mathbf{P}^{n+m}_K.$$

In other words, for $x \in X(\overline{K})$ we have

$$H_{\mathscr{L}_2}(x) = \frac{1}{[K' : K]} \sum_{v'} \max_i \log \|\sigma_i(x)\|_{v'}$$

and

$$H_{\mathscr{L}_1}(x) = \frac{1}{[K' : K]} \sum_{v'} \max(\max_i \log \|s\sigma_i(x)\|_{v'}, \max_j \log \|\tau_j(x)\|_{v'})$$

where K'/K is a finite subextension of $\overline{K}$ such that $x \in X(K')$ (and v' ranges over the places of K'). Here, it is understood that the norm $\|\cdot\|_{v'}$ on the 1-dimensional K'-vector space $\mathscr{L}_i(x) \otimes_{K(x)} K'$ is defined using a *single* choice of K'-basis e_i; the specific choice doesn't matter due to the product formula; this is the same calculation used to justify the homogeneity of the standard height on projective spaces. We likewise take the K'-basis of $\mathscr{L}(x) \otimes_{K(x)} K'$ to be $e_1 \otimes e_2^*$.

Since we are restricting attention to points $x \in X_s(\overline{K})$, we have $s(x) \neq 0$, so we can scale by $\|s(x)\|_{v'}$ throughout to get

$$H_{\mathscr{L}_1}(x) = \frac{1}{[K' : K]} \sum_{v'} \max(\max_i \log \|\sigma_i(x)\|_{v'}, \max_j \log \|\tau_j(x)/s(x)\|_{v'}) \geq H_{\mathscr{L}_2}(x).$$

Thus, $H_{\mathscr{L}_1}(x) - H_{\mathscr{L}_2}(x) \geq 0$.