# Math 248B. Homework 7

1. Let $E \to M$ be an elliptic curve over a complex manifold $M$, $C \subset E$ a closed $M$-subgroup that is a degree-$n$ covering of $M$ with cyclic fibers, and $\phi : E \to E' := E/C$ the natural isogeny. Let $\phi^* : \omega_{E'/M} \to \omega_{E/M}$ be the natural map of line bundles (arising from pullback on relative 1-forms over $M$). Prove that the diagram

$$\begin{tikzcd} \omega_{E'/M}^{\otimes 2} \arrow[r, "(1/n)(\phi^*)^{\otimes 2}"] \arrow[d, "\mathrm{KS}_{E'/M}"'] & \omega_{E/M}^{\otimes 2} \arrow[d, "\mathrm{KS}_{E/M}"] \\ \Omega^1_M \arrow[r, equal] & \Omega^1_M \end{tikzcd}$$

commutes. (Hint: By base change and working locally on the base, reduce to the case of the Weierstrass family over $\mathbf{C} - \mathbf{R}$ and the cyclic subgroup generated by $1/n$ in $\mathscr{E}_\tau = \mathbf{C}/\Lambda_\tau$. Then compute explicitly.)

2. Let $M$ be a connected topological manifold $M$ with pure dimension $n$. Recall that there is an *orientation sheaf* $o_M$ which keeps track of local orientations (and has $m$-stalk $o_{M,m}$ canonically isomorphic to $\mathrm{H}_n(M, M - \{m\}; \mathbf{Z})$). This is a local system of finite free $\mathbf{Z}$-modules of rank 1, and $o_M(M) \neq 0$ if and only if $M$ is orientable (in which case $o_M \simeq \underline{\mathbf{Z}}$).

(i) Prove that canonically $\mathrm{H}^n_c(M, o_M) \simeq \mathbf{Z}$, and that if $M$ is a complex manifold then canonically $o_M \simeq \underline{\mathbf{Z}(n)}$ so that the $\mathbf{C}$-scalar extension of $\mathrm{H}^n_c(M, o_{M,\mathbf{C}}) \simeq \mathbf{C}$ is $(1/2\pi i)^n \int_{M,i}$.

(ii) Read through your favorite proof of Poincaré duality relating cohomology to the dual of compactly supported cohomology (with field coefficients), and check that the proof carries over to show that if $\mathscr{F}$ is a local system of finite-dimensional vector spaces over a field $k$ then the cup product pairing

$$\mathrm{H}^j_c(M, \mathscr{F}) \times \mathrm{H}^{n-j}(M, \mathscr{F}^* \otimes o_M) \to \mathrm{H}^n_c(M, o_{M,k}) \simeq k$$

identifies the cohomology with the linear dual of the compactly supported cohomology.

(iii) Fix $k \geq 2$ and $\Gamma$ as in the setup for the Eichler-Shimura morphism $\mathrm{ES}_\Gamma$, and let $\mathscr{F} = \mathrm{R}^1 f_{\Gamma *}(\mathbf{Z})$. Using the Weil pairing on elliptic curves, produce a canonical symplectic form $\mathscr{F} \times \mathscr{F} \to \underline{\mathbf{Z}}(1)$, and thereby construct a canonical symplectic form on $\mathrm{Sym}^r(\mathscr{F})$ valued in $\underline{\mathbf{Z}}(r)$. Combining this with (ii) and the canonical isomorphism $\mathbf{C}(1) \simeq \mathbf{C}$ defined by multiplication, construct a perfect bilinear form on $\widetilde{\mathrm{H}}^1(Y_\Gamma, \mathrm{Sym}^{k-2}(\mathscr{F})_{\mathbf{C}})$ (whose symmetry depends on the parity of $k$).

(iv) Using excision arguments, prove that if $M$ has a finite good cover (so on those opens and their iterated overlaps all local systems are constant) then for any local system $\mathscr{F}$ of finitely generated modules of a noetherian ring $R$, $\mathrm{H}^i(M, \mathscr{F})$ is a finitely generated $R$-module and vanishes for large $i > \dim M$. In the special case that $R = k$ is a field and the fibers of $\mathscr{F}$ have dimension $d$, prove that $\chi(\mathscr{F}) = d\chi(M)$. How about compactly supported cohomology?

3. Fix $N \geq 4$ $k \geq 2$, and a prime $p$ (allowing $p|N$). Consider the finite-degree analytic coverings $\pi_1, \pi_2 : Y_1(N, p) \rightrightarrows Y_1(N)$.

(i) Define suitable trace maps on $\mathrm{H}^1_c$'s and $\mathrm{H}^1$'s so as to define a "Hecke operator" $T_p^{\mathrm{top}} \in \mathrm{End}_{\mathbf{Z}}(\mathrm{H}^1(Y_\Gamma, \mathscr{F}_{k-2}))$ that preserves the submodule $\widetilde{\mathrm{H}}_1$.

(ii) Using Exercise 1, prove that $\mathrm{ES}_\Gamma$ intertwines $(T_p^{\mathrm{top}})_{\mathbf{C}}$ with the direct sum of the classical $T_p$ and its conjugate on

$$S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} = \mathbf{C} \otimes_{\mathbf{R}} S_k(\Gamma_1(N)).$$

How about the diamond operators?

(iii) Deduce that there is a single number field whose ring of integers contains all eigenvalues on $S_k(\Gamma_1(N))$ of all Hecke operators $T_p$. Explain this more concrete when $k = 2$. Can you deduce these properties from the holomorphic theory alone (without intervention of integral topological cohomology structures)?