Math 248B. Homework 6

1. For any integer $N \geq 1$ and a prime p, define a $\Gamma_1(N,p)$-*structure* on an elliptic curve $E \to M$ to be a pair (P,C) consisting of a holomorphic section $P \in E[N](M)$ with exact order N on fibers and a $\Gamma_0(p)$-structure C (i.e., closed M-subgroup $C \subset E$ that is a degree-p finite analytic covering of M) such that $\langle P \rangle \cap C = 0$ inside of E. (Equivalently, $\langle P(m) \rangle \cap C_m = 0$ inside E_m for all $m \in M$.) Here, $\langle P \rangle$ denotes the M-subgroup $(\mathbf{Z}/N\mathbf{Z}) \times M \hookrightarrow E$ defined by $(j,m) \mapsto jP(m)$. The notions of isomorphism and base change for such triples (E,P,C) is defined in the evident manner.

(i) Define the subgroup $\Gamma_1(N,p) \subseteq \mathrm{SL}_2(\mathbf{Z})$ to be $\Gamma_1(N) \cap \Gamma_0(p)$ if $p \nmid N$, and $\Gamma_1(N) \cap \Gamma^0(p)$ if $p|N$ (where $\Gamma^0(p)$ is the transpose of $\Gamma_0(p)$). For a classical elliptic curve E and a $\Gamma_1(N,p)$-structure (P,C) on E, prove that when $p \nmid N$ there exists $\tau \in \mathbf{C} - \mathbf{R}$ such that $(E,P,C) \simeq (\mathbf{C}/\Lambda_\tau, 1/N, \langle 1/p \rangle)$, and that when $p|N$ there exists $\tau \in \mathbf{C} - \mathbf{R}$ such that $(E,P,C) \simeq (\mathbf{C}/\Lambda_\tau, 1/N, \langle \tau/p \rangle)$.

In all cases prove that τ can be found in a fixed choice of connected component of $\mathbf{C} - \mathbf{R}$, and that as such it is unique up to precisely the $\Gamma_1(N,p)$-action.

(ii) If $p \nmid N$ then define $G\Gamma_1(N,p) = G\Gamma_1(N) \times G\Gamma_0(p) \subseteq \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \times \mathrm{GL}_2(\mathbf{Z}/p\mathbf{Z}) = \mathrm{GL}_2(Z/Np\mathbf{Z})$, and if $p|N$ then define $G\Gamma_1(N,p) = G\Gamma_1(N) \cap G\Gamma^0(p) \subseteq \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, where $G\Gamma^0(p) = \{\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)\}$. Prove that $\Gamma_1(N,p)$ is the intersection of $\mathrm{SL}_2(\mathbf{Z})$ with the preimage of $G\Gamma_1(N,p)$ under $\mathrm{GL}_2(\mathbf{Z}) \to \mathrm{GL}_2(\mathbf{Z}/Np\mathbf{Z})$ when $p \nmid N$ (resp. $\mathrm{GL}_2(\mathbf{Z}) \to \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ when $p|N$), and in all cases naturally identify $G\Gamma_1(N,p)$-structures on E with $\Gamma_1(N,p)$-structures as defined above. Prove rigidity holds if and only if $N \geq 4$.

(iii) Rigorously prove that the moduli problem of isomorphism classes of $\Gamma_1(N,p)$-structures admits a coarse moduli space $Y_1(N,p)$, fine when $N \geq 4$, and give an explicit description of it as a quotient of the i-component of $\mathbf{C} - \mathbf{R}$ for a choice of i (and specify a universal structure over it when $N \geq 4$).

2. Fix $N \geq 4$ and let $E \to Y_1(N)$ denote the universal elliptic curve. Let $\omega = \omega_{E/Y_1(N)}$. Choose $i = \sqrt{-1} \in \mathbf{C}$, and thereby identify $Y_1(N)$ with a quotient of the i-component $\mathfrak{h}_i$ of $\mathbf{C} - \mathbf{R}$ as in class. In this way we identified $\mathrm{H}^0(Y_1(N), \omega^{\otimes k})$ with the space of holomorphic functions $f : \mathfrak{h}_i \to \mathbf{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma_1(N,p)$.

(i) Prove by fibral computations with universal structures that for *any* prime p (allowing $p|N$) the endomorphism of $\mathrm{H}^0(Y_1(N), \omega^{\otimes k})$ defined in class by the pullback/trace method with $Y_1(N,p)$ coincides with the classical operation pT_p, and do likewise with the diamond operators $\langle a \rangle$ for $a \in (\mathbf{Z}/N\mathbf{Z})^\times$.

(ii) Fix a prime p and let $(E_Y, P_Y, C_Y) \to Y = Y_1(N,p)$ be the universal object and $\varphi : E_Y \to E'_Y = E_Y/C_Y$ the natural isogeny. Let $\mathrm{KS}_N : \omega^{\otimes 2} \to Y_1(N)$ be the Kodaira-Spencer map from HW5 (an isomorphism). Consider the pullback map $(1/p)\varphi^* : \omega_{E'_Y/Y} \to \omega_{E_Y/Y}$. Prove that the resulting composite map

$$\pi_2^*(\omega^{\otimes 2}) \simeq \omega_{E'_Y/Y}^{\otimes 2} \to \omega_{E_Y/Y}^{\otimes 2} \simeq \pi_1^*(\omega^{\otimes 2})$$

intertwines $\pi_2^*(\mathrm{KS}_N)$ and $\pi_1^*(\mathrm{KS}_N)$ via the natural identifications of $\pi_j^*(\Omega^1_{Y_1(N)})$ with Ω^1_Y.

3. Let K be a compact open subgroups of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ such that K-structures are rigid, and choose $g \in \mathrm{Mat}_2(\mathbf{Z}) \cap \mathrm{GL}_2(\mathbf{Q}) \subset \mathrm{GL}_2(\mathbf{A}_f)$. Let $K' = K \cap g^{-1}Kg \subseteq \mathrm{GL}_2(\widehat{\mathbf{Z}})$ (note that $K', gK'g^{-1} \subseteq K$), and assume that $\det(K') = \widehat{\mathbf{Z}}^\times$. Let ω_K be the line bundle on Y_K arising from the universal elliptic curve (in the isogeny category). Fix a choice of i and identity Y_K as a quotient of $\mathfrak{h}_i$ modulo $\Gamma := K \cap \mathrm{SL}_2(\mathbf{Z})$.

(i) Identify $\mathrm{H}^0(Y_K, \omega_K^{\otimes k})$ with the space of holomorphic $f : \mathfrak{h}_i \to \mathbf{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

(ii) Using the finite maps $J_{K',K}(g), J_{K',K}(1) : Y_{K'} \rightrightarrows Y_K$, adapt the pullback/trace method to construct an endomorphism of $\mathrm{H}^0(Y_K, \omega_K^{\otimes k})$, and prove that it coincides with the classical operator $[\Gamma g \Gamma]_k$ up to a scaling factor depending on $\det g$ and k.

4. Let $f : E \to M$ be an elliptic curve over a complex manifold. Prove the existence and uniqueness of a map of $\mathscr{O}_M$-modules $\omega_{E/M} \to \mathscr{O}_M \otimes_{\underline{\mathbf{C}}} \mathrm{R}^1 f_*(\underline{\mathbf{C}})$ inducing the classical map $\mathrm{H}^0(E_m, \Omega^1) \to \mathrm{H}^1(E_m, \mathbf{C})$ on fibers. (Bonus: can you construct this as a connecting map in a long exact sequence of higher direct images, at least *up to a sign*?)