1. For any integer $N \geq 1$ and a prime $p$, define a $\Gamma_1(N,p)$-structure on an elliptic curve $E \to M$ to be a pair $(P,C)$ consisting of a holomorphic section $P \in E[N](M)$ with exact order $N$ on fibers and a $\Gamma_0(p)$-structure $C$ (i.e., closed $M$-subgroup $C \subseteq E$ that is a degree-$p$ finite analytic covering of $M$) such that $\langle P \rangle \cap C = 0$ inside of $E$. (Equivalently, $\langle P(m) \rangle \cap C_m = 0$ inside $E_m$ for all $m \in M$.) Here, $(P)$ denotes the $M$-subgroup $(\mathbb{Z}/N\mathbb{Z}) \times M \to E$ defined by $(j,m) \mapsto jP(m)$. The notions of isomorphism and base change for such triples $(E,P,C)$ is defined in the evident manner.

(i) Define the subgroup $\Gamma_1(N,p) \subseteq \text{SL}_2(\mathbb{Z})$ to be $\Gamma_1(N) \cap \Gamma_0(p)$ if $p \nmid N$, and $\Gamma_1(N) \cap \Gamma_0^0(p)$ if $p|N$ (where $\Gamma_0^0(p)$ is the transpose of $\Gamma_0(p)$). For a classical elliptic curve $E$ and a $\Gamma_1(N,p)$-structure $(P,C)$ on $E$, prove that when $p
mid N$ there exists $\tau \in C - \mathbb{R}$ such that $(E,P,C) \simeq (C/\Lambda_r,1/N,(1/p))$, and that when $p|N$ there exists $\tau \in C - \mathbb{R}$ such that $(E,P,C) \simeq (C/\Lambda_r,1/N,\langle \tau/p \rangle))$.

In all cases prove that $\tau$ can be found in a fixed choice of connected component of $C - \mathbb{R}$, and that as such it is unique up to precisely the $\Gamma_1(N,p)$-action.

(ii) If $p \nmid N$ then define $G_1(N,p) = G_1(N) \times G_0(p) \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}/p\mathbb{Z}) = \text{GL}_2(\mathbb{Z}/Np\mathbb{Z})$, and if $p|N$ then define $G_1(N,p) = G_1(N) \cap G_0^0(p) \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, where $G_0^0(p) = \{(*_0)_0 \}$. Prove that $\Gamma_1(N,p)$ is the intersection of $\text{SL}_2(\mathbb{Z})$ with the preimage of $G_1(N,p)$ under $\text{GL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/Np\mathbb{Z})$ when $p \nmid N$ (resp. $\text{GL}_2(\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ when $p|N$), and in all cases naturally identify $G_1(N,p)$-structures on $E$ with $\Gamma_1(N,p)$-structures as defined above. Prove rigidity holds if and only if $N \geq 4$.

(iii) Rigorously prove that the moduli problem of isomorphism classes of $\Gamma_1(N,p)$-structures admits a coarse moduli space $Y_1(N,p)$, fine when $N \geq 4$, and give an explicit description of it as a quotient of the $i$-component of $C - \mathbb{R}$ for a choice of $i$ (and specify a universal structure over it when $N \geq 4$).

2. Fix $N \geq 4$ and let $E \to Y_1(N)$ denote the universal elliptic curve. Let $\omega = \omega_{E/Y_1(N)}$. Choose $i = \sqrt{-1} \in \mathbb{C}$, and thereby identify $Y_1(N)$ with a quotient of the $i$-component $h_i$ of $C - \mathbb{R}$ as in class. In this way we identified $H^0(Y_1(N),\omega \otimes k)$ with the space of holomorphic functions $f : h_i \to C$ such that $f|_{h_1} = f$ for all $\gamma \in \Gamma_1(N,p)$.

(i) Prove by fibral computations with universal structures that for any prime $p$ (allowing $p|N$) the endomorphism of $H^0(Y_1(N),\omega \otimes k)$ defined in class by the pullback/trace method with $Y_1(N,p)$ coincides with the classical operation $pT_p$, and do likewise with the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

(ii) Fix a prime $p$ and let $(E_Y,P_Y,C_Y) \to Y = Y_1(N,p)$ be the universal object and $\varphi : E_Y \to E_Y' = E_Y/C_Y$ the natural isogeny. Let $K_{S,N} : \omega^2 \to Y_1(N)$ be the Kodaira-Spencer map from HW5 (an isomorphism). Consider the pullback map $(1/p)\varphi^* : \omega^2_{E_Y/Y} \to \omega^2_{E_Y'/Y}$. Prove that the resulting composite map

$$\pi_1^2(\omega^2_{E_Y/Y}) \to \omega^2_{E_Y'/Y} \to \omega^2_{E_Y'/Y} \simeq \pi_1^2(\omega^2_{E_Y/Y})$$

intertwines $\pi_1^2(K_{S,N})$ and $\pi_1^2(K_{S,N})$ via the natural identifications of $\pi_1^2(\Omega^1_{Y_1(N)})$ with $\Omega^1_{Y}$.

3. Let $K$ be a compact open subgroups of $\text{GL}_2(\mathbb{Z})$ such that $K$-structures are rigid, and choose $g \in \text{Mat}_2(\mathbb{Z}) \cap \text{GL}_2(\mathbb{Q}) \subseteq \text{GL}_2(\mathbb{A}_f)$. Let $K' = K \cap g^{-1}Kg \subseteq \text{GL}_2(\mathbb{Z})$ (note that $K', gK'g^{-1} \subseteq K$), and assume that $\det(K') = \mathbb{Z}$. Let $\omega_K$ be the line bundle on $Y_K$ arising from the universal elliptic curve (in the isogeny category). Fix a choice of $i$ and identify $Y_K$ as a quotient of $h_i$ modulo $\Gamma := K \cap \text{SL}_2(\mathbb{Z})$.

(i) Identify $H^0(Y_K,\omega \otimes k)$ with the space of holomorphic $f : h_i \to C$ such that $f|_{h_1} = f$ for all $\gamma \in \Gamma$.

(ii) Using the finite maps $J_{K',K}(g) : Y_{K'} \to Y_K$, adapt the pullback/trace method to construct an endomorphism of $H^0(Y_K,\omega \otimes k)$, and prove that it coincides with the classical operator $[\Gamma g \Gamma]_k$ up to a scaling factor depending on $\det g$ and $k$.

4. Let $f : E \to M$ be an elliptic curve over a complex manifold. Prove the existence and uniqueness of a map of $\mathcal{O}_M$-modules $\omega_{E/M} \to \mathcal{O}_M \otimes _\mathbb{C} \mathbb{R}^1f_*(\mathcal{O}_C)$ inducing the classical map $H^0(E_m,\Omega^1) \to H^1(E_m,\mathbb{C})$ on fibers. (Bonus: can you construct this as a connecting map in a long exact sequence of higher direct images, at least up to a sign?)