Math 248B. Vector bundles modulo relative lattices

1. Motivation

Let $M$ be a complex manifold, $V \to M$ a holomorphic vector bundle of rank $g > 0$, and $L \to M$ a local system of finite free $\mathbb{Z}$-modules of rank $2g$ (see Exercise 2 in HW1). Suppose there is given a map of $M$-groups $j : L \to V$. We are interested in situations for which each fibral map $\mathbb{Z}^{2g} \cong L_m \to V_m \cong \mathbb{C}^g$ is a co-compact lattice (with inclusion depending on $m \in M$), and gluing the quotients $V_m/L_m$ into a “total space” over $M$ in a nice way. The case of most interest to us is $g = 1$, and in effect this is all a vast generalization of the Weierstrass construction

$$\mathbb{Z}^2 \times (\mathbb{C} - \mathbb{R}) \to \mathbb{C} \times (\mathbb{C} - \mathbb{R})$$

over $\mathbb{C} - \mathbb{R}$ defined by $((m, n), \tau) \mapsto (m \tau + n, \tau)$. As a warm-up, we present the higher-rank generalization.

Fix $i = \sqrt{-1} \in \mathbb{C}$ (to define “imaginary part”), and let $\mathfrak{h}_{g,i} \subset \text{Mat}_{g \times g}(\mathbb{C})$ be the subset of $g \times g$ matrices $Z$ that are symmetric and have imaginary part $\text{im}(Z)$ which is positive-definite (in the sense that the corresponding symmetric bilinear form over $\mathbb{R}$ is positive-definite). This is the the Siegel upper half-space; for $g = 1$ it is the usual upper half-plane (the connected component of $\mathbb{C} - \mathbb{R}$ containing the chosen $i$).

Obviously $\mathfrak{h}_{g,i}$, is an open locus in the vector space of symmetric $g \times g$ matrices, so it is naturally a complex manifold. Less obvious is the fact (which is trivial for $g = 1$) that $\mathfrak{h}_{g,i}$ is connected; we will not need it, so we do not prove it.

Remark 1.1. In general, $\mathfrak{h}_{g,i} \subset \text{GL}_g(\mathbb{C})$. For $g = 1$ this is the obvious assertion that the half-planes in $\mathbb{C} - \mathbb{R}$ lie in $\mathbb{C}^\times$. In general we argue as follows. Pick $Z \in \mathfrak{h}_{g,i}$. We aim to prove that ker $Z = 0$ (so $Z$ is invertible). Using the standard hermitian form $(\bar{z}, \bar{w}) = \sum z_j \bar{w}_j$ on $\mathbb{C}^g$,

$$\langle (Z - \bar{Z})(v), v \rangle = \langle Zv, v \rangle - \langle \bar{Z}v, v \rangle = \langle Zv, v \rangle - \langle v, Zv \rangle,$$

where the final equality uses the symmetry of $Z$. Thus, if $Zv = 0$ then the sesquilinear form $B(x, y) = \langle (Z - \bar{Z})(x), y \rangle$ on $\mathbb{C}^g$ satisfies $B(v, v) = 0$. But $Z - \bar{Z} = 2i\text{im}(Z)$ with $\text{im}(Z)$ positive-definite, so $B = 2iH$ where the sesquilinear form $H$ arises from an inner product on $\mathbb{R}^g$. In particular, $H$ is hermitian, so the vanishing of $H(v, v)$ forces $v = 0$.

Consider the map of $\mathfrak{h}_{g,i}$-groups $j : \mathbb{Z}^{2g} \times \mathfrak{h}_{g,i} \to \mathbb{C}^g \times \mathfrak{h}_{g,i}$ defined by $(\bar{m}, Z) \mapsto ((Z - 1_g)\bar{m}, Z)$. (For $g = 1$, writing $\bar{m} = (\bar{m})$ recovers the Weierstrass construction.) We claim that for every $Z \in \mathfrak{h}_{g,i}$, the map on $Z$-fibers is a co-compact lattice inclusion. In other words, we claim that $j_Z : \mathbb{Z}^{2g} \oplus \mathbb{Z}^2 \to \mathbb{C}^g$ is injective and a co-compact lattice. To see this, the following lemma is convenient:

Lemma 1.2. A homomorphism $j : L_0 \to V_0$ from a finite free $\mathbb{Z}$-module of rank $2g$ to a $g$-dimensional $C$-vector space is a co-compact lattice inclusion if and only if the natural $C$-linear map of $C$-vector spaces $C \otimes_{\mathbb{Z}} L_0 \to V_0 \oplus \overline{V}_0$ defined by $c \otimes \lambda \mapsto (c\lambda(\lambda), c\lambda(\bar{\lambda}))$ is an isomorphism, where $\overline{V}_0 = C \otimes_{\sigma, C} V_0$ is the conjugate space (scalar extension by complex conjugation $\sigma$) and $\sigma := 1 \otimes v$ for $v \in V_0$. Equivalently, upon choosing bases $L_0 \cong \mathbb{Z}^{2g}$ and $V_0 \cong \mathbb{C}^g$ to identify $j$ with a $g \times 2g$ matrix $(A \ B)$ for $A, B \in \text{Mat}_{g \times g}(\mathbb{C})$, the necessary and sufficient condition is that the matrix

$$\begin{pmatrix} A & B \\ \overline{A} & \overline{B} \end{pmatrix} \in \text{Mat}_{2g \times 2g}(\mathbb{C})$$

is invertible.

Beware that an abstract $C$-vector space does not have an intrinsic “complex conjugation” operator (semilinear over complex conjugation on $C$), so $V_0$ and $\overline{V}_0$ are not naturally $C$-linearly isomorphic. The natural map $V_0 \to V_0$ defined by $v \mapsto \tau$ is merely semilinear over the complex conjugation on $C$. Likewise, if we choose a $C$-basis $\{v_k\}$ of $V_0$ and use $\{v_k\}$ as a $C$-basis of $\overline{V}_0$ then the composition $C^g = V_0 \to \overline{V}_0 = C^g$ is coordinate-wise complex conjugation.

Proof. The co-compact lattice condition is precisely that the $R$-linear map $j_R : R \otimes_{\mathbb{Z}} \mathbf{L}_0 \to V_0$ is an isomorphism, and this is equivalent to the isomorphism condition after applying scalar extension by $R \to C$. But $C \otimes_R V_0 = (C \otimes_R C) \otimes_C V_0$ with $C \otimes_R C \cong C \times C$ as $C$-algebras via $a \otimes b \mapsto (ab, \overline{ab})$ (we view $C \otimes_R C$ as
a $C$-algebra using the left tensor structure). Hence, $C \otimes_R V_0$ is identified as a $C$-vector space with $V_0 \oplus \overline{V}_0$, and in this way the $C$-linear scalar extension of $j_R$ is identified with the map introduced in the statement of the lemma. The matrix interpretation is immediate upon identifying the $2g \times 2g$ matrix as computing this $C$-linear map relative to the $C$-basis of $C \otimes_Z L_0$ coming from the chosen $Z$-basis of $L_0$ and the $C$-basis of $V_0 \oplus \overline{V}_0$ coming from the chosen basis of $V_0$ and the corresponding conjugate basis of $\overline{V}_0$. ■

By this lemma, to prove that $j_Z$ is a co-compact lattice inclusion it is equivalent to prove invertibility of the $2g \times 2g$ matrix

$$
\begin{pmatrix}
Z & 1_g \\
\overline{Z} & 1_g
\end{pmatrix},
$$

Subtracting the bottom $g \times 2g$ block from the upper one gives

$$
\begin{pmatrix}
Z - \overline{Z} & 0 \\
\overline{Z} & 1_g
\end{pmatrix},
$$

so its invertibility is equivalent to that of the $g \times g$ matrix $Z - \overline{Z}$. This is exactly $2i\text{im}(Z)$, and by definition of $h_{g,i}$, the real matrix $\text{im}(Z)$ corresponds to a positive-definite symmetric bilinear form over $R$. Such bilinear forms are non-degenerate, so their associated matrix is always invertible (over $R$ or over $C$, which come to the same thing). This proves that the $h_{g,i}$-group map $j$ is a co-compact lattice inclusion on all fibers over $h_{g,i}$.

2. A Quotient Construction: $C^\infty$-Aspects

Now consider a general map of $M$-groups $j : L \to V$ where $V$ is a rank-$g$ vector bundle over a complex manifold $M$ and $L$ is a local system of rank-$2g$ finite free $Z$-modules. We are interested in the cases when the fibral maps $j_m : L_m \to V_m$ are all co-compact lattice inclusions, but we first note that this property of the fibral map is open on the base:

**Lemma 2.1.** If $j_{m_0}$ is a co-compact lattice inclusion for some $m_0 \in M$, then the same holds for $j_m$ for all $m$ in an open neighborhood of $m_0$ in $M$.

**Proof.** The problem is local around $m_0$, so we may shrink to acquire local frames for $L$ and $V$. That is, we may arrange that the local system $L$ is split and the vector bundle $V$ is free, which is to say that there is an $M$-group isomorphism $L \simeq Z^{2g} \times M$ and a vector bundle isomorphism $V \simeq C^g \times M$ over $M$. Let $e_1, \ldots, e_{2g} \in L(M)$ correspond to the splitting of $L$ (i.e., $\{e_k(m)\}$ is the resulting basis of the fiber $L_m$ for each $m \in M$), so the $M$-group map $j$ amounts to the specification of the holomorphic sections $s_k = j \circ e_k : M \to V$. Relative to the chosen global frame $\{v_1, \ldots, v_g\}$ in $V(M)$, we have $s_k = \sum a_{hk} v_h$ in $V(M)$ for holomorphic $a_{hk} : M \to C$. Thus, $j$ corresponds to the $g \times 2g$ matrix $T := (a_{hk})$ of holomorphic functions, and by Lemma 1.2 the hypothesis at $m_0$ is that the $2g \times 2g$ matrix $(T)$ of continuous $C$-valued functions is invertible at $m_0$. It is therefore invertible in a neighborhood of $m_0$. ■

Now suppose that all fibral maps $j_m$ are co-compact lattice inclusions. We impose an equivalence relation on the set $V$ as follows: $v \sim v'$ if $v$ and $v'$ lie over the same point $m \in M$ and if $v - v' \in L_m$ inside of $V_m$. In other words, we impose the fibral equivalence relation of congruence modulo the lattice $L_m$ in $V_m$ for every $m \in M$. Define the topological space $V/L$ to be the quotient of $V$ modulo $\sim$, so there is are natural continuous map $V \to V/L$ and $V/L \to M$. We then have the following purely topological result:

**Proposition 2.2.** The map $\pi : V \to V/L$ is a covering space, and $V/L \to M$ is proper. There is a unique $C^\infty$-structure relative to which $\pi$ is a local $C^\infty$-isomorphism, and then $V/L \to M$ is a submersion and even a split $C^\infty$ fiber bundle with fiber $(S^1)^g$.

The map $\pi$ is a $C^\infty$ quotient in the sense that any $C^\infty$ map $V \to N$ that is $\sim$-invariant uniquely factors through $\pi$ via a $C^\infty$-map $V/L \to N$.

The careful reader will see that the proof is nothing but a slick generalization of the special case in class for $g = 1$ and the Weierstrass construction over $M = C - R$. 

Proof. We may work locally over $M$, so as in the proof of Lemma 2.1 we can assume that $L = \mathbb{Z}^{2g} \times M$ and $V = \mathbb{C}^g \times M$ with $j$ corresponding to a $g \times 2g$ matrix $(a_{hk})$ of holomorphic functions $T = (a_{hk})$ such that the $2g \times 2g$ matrix
\[
\begin{pmatrix}
T(m) \\
T(m)
\end{pmatrix} \in \text{Mat}_{2g \times 2g}(\mathbb{C})
\]
is invertible for each $m \in M$. In particular, the top $g \times 2g$ matrix $T(m)$ is surjective as a linear map $\mathbb{C}^{2g} \to \mathbb{C}^g$, so by equality of row rank and column rank there is an invertible $g \times g$ submatrix. Working locally around some $m_0 \in M$, we may rearrange the order of the trivialization of $L$ so that the right $g \times g$ submatrix of $T(m_0)$ is invertible. By shrinking around $m_0$ we can then assume that the right $g \times g$ submatrix of $T(m)$ is invertible for all $m \in M$. In other words, $j(e_{g+1}), \ldots, j(e_{2g}) \in V(M)$ is a global frame, where $\{e_1, \ldots, e_{2g}\}$ in $L(M)$ is the chosen trivialization of $L \to M$.

Writing $T = (A \ B)$ with $g \times g$ matrices $A$ and $B$ whose entries are holomorphic functions, we have arranged that $B$ is invertible, so by modifying the initial isomorphism $V \simeq \mathbb{C}^g \times M$ via $B^{-1}$ we can arrange that $T = (1 \ g)$ for some holomorphic map $Z : M \to \text{Mat}^{g \times g}(\mathbb{C})$. Consider the $C^\infty$-map $\overline{Z} : M \to \text{Mat}_{g \times g}(\mathbb{C})$.

By Lemma 1.2, the fibral lattice condition says exactly that the $C^\infty$-map
\[
\begin{pmatrix}
Z \\
\overline{Z}
\end{pmatrix} \in \text{Mat}_{2g \times 2g}(\mathbb{C})
\]
is valued in $\text{GL}_{2g}(\mathbb{C})$, or equivalently (by subtracting the bottom $g \times 2g$ block from the top one) that $\text{im}(Z) = Z - \overline{Z}$ is valued in $\text{GL}_g(\mathbb{C})$. That is, $\text{im}(Z)$ is invertible, which is to say $Z = A + iB$ for continuous maps $A, B : M \to \text{Mat}_g(\mathbb{R})$ with $B$ valued in $\text{GL}_g(\mathbb{R})$. Consider the $C^\infty$-map $\mathbb{C}^g \times M \to \mathbb{C}^g \times M$ over $M$ defined by $(x + iy, m) \mapsto (x + (A(m) + iB(m))y, m) = ((x + A(m)y) + iB(m)y, m)$. This is clearly a $C^\infty$-isomorphism, as the inverse is $(u + iv, m) \mapsto ((u - A(m)B(m)^{-1}v) + iB(m)^{-1}v, m)$. This isomorphism carries $j : \mathbb{Z}^{2g} \times M \to \mathbb{C}^g \times M$ over to the “constant” inclusion $\mathbb{Z}[i]^g \times M \to \mathbb{C}^g \times M$. Thus, topologically we see that $V/L$ over $M$ is precisely $(\mathbb{C}/\mathbb{Z}[i])^g \times M$ with $V \to V/L$ going over to the topological product of $M$ against the $g$-fold product of the natural quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}[i] = S^1 \times S^1$ that is clearly a covering space map. Likewise, $V/L \to M$ is topologically the projection $(\mathbb{C}/\mathbb{Z}[i])^g \times M \to M$ which is visibly proper.

Using the natural $C^\infty$-structure on $(\mathbb{C}/\mathbb{Z}[i])^g \times M$ then equips $V/L$ with a $C^\infty$-structure relative to which the quotient map $\pi$ is a local $C^\infty$-isomorphism since $\mathbb{C} \to \mathbb{C}/\mathbb{Z}[i]$ has that property. This $C^\infty$-structure on $V/L$ is uniquely determined by the property that $\pi$ is a local $C^\infty$-isomorphism, as this latter map is a surjective local homeomorphism. (The real issue is the existence of such a $C^\infty$-structure, which we proved already.) The preceding calculations using $C^\infty$-isomorphisms then also show that $V/L$ admits a $C^\infty$-isomorphism to $(\mathbb{C}/\mathbb{Z}[i])^g \times M = (S^1)^{2g} \times M$ over $M$.

Finally, we address the $C^\infty$-quotient property of $\pi$. This amounts to proving that a $C^\infty$-map $h : \mathbb{C}^g \times M \to N$ that is invariant by the $\mathbb{Z}[i]^g$-translation on $\mathbb{C}^g$ factors through $\pi : \mathbb{C}^g \times M \to (\mathbb{C}/\mathbb{Z}[i])^g \times M$ via a $C^\infty$-map $\overline{h} : (\mathbb{C}/\mathbb{Z}[i])^g \times M \to N$. There is certainly a unique continuous factorization since $\pi$ is a proper surjective map, and $\overline{h}$ is $C^\infty$ because its composition with the surjective local $C^\infty$-isomorphism $\pi$ is the map $h$ which is assumed to be $C^\infty$.

3. A QUOTIENT CONSTRUCTION: COMPLEX-ANALYTIC ASPECTS

Before we enhance $V/L$ to a complex manifold in a useful way, we record a fibral isomorphism criterion:

Lemma 3.1. Let $X, Y \to S$ be $C^\infty$ submersions between $C^\infty$ manifolds, and let $f : X \to Y$ be a $C^\infty$ map over $S$ such that the induced map $f_* : T_x S \to T_y S$ between fibers over each $s \in S$ is a $C^\infty$ isomorphism. Then $f$ is a $C^\infty$ isomorphism. The same holds in the category of complex manifolds, using holomorphic maps.

Proof. Clearly $f$ is bijective, so it suffices to prove that it is a local isomorphism. By the inverse function theorem, it is equivalent to prove that for each $x \in S$ the derivative map $df(x) : T_x(X) \to T_{f(x)}(Y)$ is an isomorphism. If $s \in S$ is that point over which $x$ and $f(x)$ lie, the submersion property for the maps $X, Y \to S$ gives that the maps $T_x(X), T_{f(x)}(Y) \to T_s(S)$ are surjective. Moreover, by the submersion theorem, the respective kernels are identified with $T_x(X_s)$ and $T_{f(x)}(Y_s)$. By the functoriality of derivative
maps (i.e., the Chain Rule), the map \( df(x) \) commutes with the quotients maps onto \( T_x(S) \) and carries \( T_x(X) \) to \( T_{f(x)}(S) \) via \( d(f_s)(x) \). That is, we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T_x(X) & \longrightarrow & T_x(S) & \longrightarrow & 0 \\
\downarrow {df(x)} & & \downarrow {df(x)} & & \downarrow {df(x)} & & \\
0 & \longrightarrow & T_{f(x)}(Y) & \longrightarrow & T_{f(x)}(S) & \longrightarrow & 0
\end{array}
\]

But \( f_s \) is an isomorphism by hypothesis, so the left arrow is an isomorphism and hence so is the middle arrow.

\[ \blacksquare \]

**Theorem 3.2.** There is a unique complex manifold structure on \( V/L \) relative to which \( \pi : V \to V/L \) is a local analytic isomorphism, and this makes \( V/L \to M \) a submersion.

The map \( \pi \) is an analytic quotient in the sense that any holomorphic map \( V \to N \) that is \(-\)-invariant uniquely factors through \( \pi \) via a holomorphic map \( V/L \to N \), and if \( h : M' \to M \) is a holomorphic map with \( j' : L' \to V' \) the base change of \( j : L \to V \) by \( h \) then the natural holomorphic map \( V'/L' \to (V/L) \times_M M' \) over \( M' \) is an isomorphism. Moreover, if \( \Lambda \to W \) is another such map over \( M \) then the resulting natural holomorphic map \( (V \oplus W)/(L \oplus \Lambda) \to (V/L) \times_M (W/\Lambda) \) over \( M \) is an isomorphism.

Most of the proof of this theorem is formal nonsense. Only at the end of the argument does a real idea emerge. In effect, the main challenge of the argument is to be rigorous.

**Proof.** The analogous result has already been proved in the \( C^\infty \) setting, so in particular the underlying \( C^\infty \)-structure from any such hypothetical complex-analytic structure on \( V/L \) would have to recover the one we have already built. Since the \( C^\infty \) uniqueness argument carries over to show that the analytic structure on \( V/L \) (if it exists!) would be uniquely characterized by requiring \( \pi \) to be a local analytic isomorphism, and the proof of the quotient mapping property carries over once we have found a complex-analytic structure on \( V/L \) to a complex manifold such that (i) \( \pi \) is holomorphic and even a local analytic isomorphism (so \( V/L \to M \) is necessarily holomorphic, as its composition with the surjective local analytic isomorphism \( \pi \) is the map \( V \to M \) that is holomorphic), (ii) the formation of this analytic structure is compatible base change on \( M \) and with fiber products over \( M \) as at the end of the statement of the theorem. (The submersion property can be verified on the underlying \( C^\infty \) manifolds, due to the consistency of \( C^\infty \) and complex-analytic tangent spaces.)

Let’s grant (i) and prove (ii) conditional on this. Consider any holomorphic map \( h : M' \to M \) and the natural holomorphic map \( \phi : V'/L' \to (V/L) \times_M M' \) over \( M' \). By Lemma 3.1, to prove this latter map is an analytic isomorphism it suffices to prove that the induced analytic map between fibers over each \( m' \in M' \) is an isomorphism. The map \( \phi \) respects the local analytic isomorphisms from \( V' = V \times_M M' \) onto both sides, so for any \( m' \in M' \) the map \( \phi_{m'} : (V'/L')_{m'} \to ((V/L) \times_M M')_{m'} = (V/L)_{h(m')} \) respects the surjective local analytic maps from \( V'_{m'} = V_{h(m')} \) onto both sides. It follows that \( \phi_{m'} \) is a local analytic isomorphism. But it is clearly bijective, and so it is an analytic isomorphism. Thus, \( \phi \) is an analytic isomorphism. Similarly, to prove that the analytic map

\[
(V \oplus W)/(L \oplus \Lambda) \to (V/L) \times_M (W/\Lambda)
\]

over \( M \) is an isomorphism, it suffices to prove that the map between fibers over each \( m \in M \) is an isomorphism. This is an analytic map

\[
((V \oplus W)/(L \oplus \Lambda))_m \to ((V/L) \times_M (W/\Lambda))_m = (V/L)_m \times (W/\Lambda)_m
\]

that respects the surjective local analytic isomorphisms from \( (V \oplus W)_m = V_m \times W_m \) onto both sides, so this fibral map is a local analytic isomorphism. But it is clearly bijective, so it is an analytic isomorphism. This completes the proof of (ii), conditional on (i).

In view of the established uniqueness of the desired complex structure (if it exists!), to construct it we may work locally over \( M \) (as the complex structures built locally over \( M \) must agree on overlaps, due to the uniqueness). Since \( \pi \) is a covering map, we can cover \( V/L \) by connected open sets \( U_k \) such that \( \pi^{-1}(U_k) \to U_k \)
is a split covering map; i.e., \( \pi^{-1}(U_k) = \Sigma_k \times U_k \) for a discrete space \( \Sigma_k \). In other words, each connected component of \( \pi^{-1}(U) \) maps homeomorphically onto \( U \). Consider an arbitrary connected open set \( U \subset V/L \) such that \( \pi^{-1}(U) \to U \) is a split covering, so every connected component of \( \pi^{-1}(U) \) maps homeomorphically onto \( U \). These components are open in \( V \) and so inherit a complex structure from \( V \). Assume (as we will prove below) that these all endow the open \( U \subset V/L \) with the same complex structure (i.e., for any two connected components \( C \) and \( C' \) of \( \pi^{-1}(U) \), the composite homeomorphism \( C \simeq U \simeq C' \) is a holomorphic isomorphism). By giving \( U \) that common complex structure, the covering map \( \pi^{-1}(U) \to U \) becomes a local analytic isomorphism. This is clearly the only complex structure on \( U \) with that property. By first letting \( U \) vary through the \( U_k \)'s and then vary through the connected components of each \( U_k \cap U_{k'} \), it follows from the uniqueness that these complex structures on the connected components of \( U_k \cap U_{k'} \) arising from \( U_k \) and \( U_{k'} \) separately must coincide. In other words, the complex structures would necessarily agree on the entire overlaps \( U_k \cap U_{k'} \) and thus globalize to a complex structure on \( V/L \) making \( \pi \) a holomorphic map and local analytic isomorphism, as desired.

So we finally come to the non-formal part of the argument, where we have to prove something specific about the complex structure on \( V \) in relation to \( L \): if \( U \subset V/L \) is a connected open set such that \( \pi^{-1}(U) \to U \) is a split covering space, then we claim that for any two connected components \( C \) and \( C' \) of \( \pi^{-1}(U) \), the composite homeomorphism \( C \simeq U \simeq C' \) is a holomorphic isomorphism. This problem is local on \( U \), in the sense that it suffices to check it over a collection of connected open sets that cover \( U \). Pick a point \( u \in U \) and let \( v \in C \) and \( v' \in C' \) be the corresponding points in \( \pi^{-1}(u) \). Let \( m \in M \) be the common image point, so \( l := v' - v \in L_m \) inside \( V_m \). Since \( L \to M \) is a local system, by working locally on \( M \) and \( m \) we can arrange that there exists a holomorphic section \( s \in L(M) \) such that \( s(m) = l \) in the fiber \( L_m \). Now \( j : L \to V \) is an \( M \)-group map, so translation on the group \( V(M) \) by \( j \circ s \in V(M) \) preserves the subgroup \( L(M) \), inducing translation by \( s \). This translation by \( j \circ s \) is a holomorphic automorphism of \( V \) over \( M \) that preserves \( L \) and carries \( v \) to \( v' \) on \( V_m \). In particular, this automorphism restricts to a holomorphic \( U \)-automorphism of \( \pi^{-1}(U) \) carrying \( v \) to \( v' \) and thus carrying the open connected component \( C \) through \( v \) onto the connected component \( C' \) through \( v' \) via a holomorphic isomorphism respecting the homeomorphic projections \( C, C' \simeq U \). It follows that the composite homeomorphism of interest \( C \simeq C' \) between open sets in \( V \) is precisely the holomorphic \( U \)-automorphism induced by translation by \( j \circ s \) on the \( M \)-group \( V \), so we are done!

**Corollary 3.3.** There is a unique \( M \)-group structure on \( V/L \) that makes \( \pi : V \to V/L \) an \( M \)-group homomorphism, and its formation is functorial in \((V,L)\) and commutes with base change on \( M \). Moreover, for ker \( \pi := V \times_{V/L,0} M \), the natural map of \( M \)-groups \( L \to \ker \pi ) \) is an isomorphism.

**Proof.** Since \( \pi \) is surjective, so \( V \times_M V \to (V/L) \times_M (V/L) \) is also surjective, such an \( M \)-group structure is certainly unique if it exists, and it inherits the functoriality and compatibility with base change from comparison with the \( M \)-group law on \( V \). To prove existence, we first note that the composite map

\[
V \oplus V = V \times_M V \xrightarrow{i} V \to V/L
\]

is invariant under the equivalence relation that defines \((V \oplus V)/(L \oplus L)\), so it uniquely factors through an \( M \)-map \((V \oplus V)/(L \oplus L) \to V/L \). But the source of this map is naturally identified with \((V/L) \times_M (V/L) \) (by the preceding Theorem), so we have constructed an \( M \)-map \( \mu : (V/L) \times_M (V/L) \to V/L \). By construction, on fibers over \( m \in M \) this is the usual group law on \( V_m/L_m \). We likewise construct the candidate \( i : V/L \simeq V/L \) over \( M \) for inversion, and comparison with \( V \) via the surjective \( \pi \) shows that \( \mu, i \), and the composite “zero section” \( M \to V \to V/L \) satisfy the \( M \)-group axioms.

Finally, to prove that the \( M \)-group map \( L \to \ker \pi \) is an isomorphism, it suffices to check that it is an isomorphism on fibers over \( M \). This reduces us to the case when \( M \) is a point, in which case the assertion is clear.

We now wish to explain the sense in which the \( M \)-group \( V/L \) behaves like a quotient sheaf over \( M \). To this end, for any \( X \to M \), let \([X]\) denote the corresponding sheaf of sets on \( M \) assigning to any open set \( U \subset M \) the set \( X(U) = \text{Hom}_M(U, X) = \text{Hom}_U(U, X_U) \) of sections of \( X_U \) over \( U \). When \( X \) is a commutative
$M$-group, then $[X]$ has a structure of abelian sheaf on $M$. By the preceding corollary, the natural diagram of abelian sheaves on $M$

$$0 \to [L] \to [V] \to [V/L]$$

is left exact, so the quotient sheaf $[V]/[L]$ over $M$ is naturally a subsheaf of $[V/L]$.

**Proposition 3.4.** The inclusion of sheaves $[V]/[L] \to [V/L]$ over $M$ is an equality.

**Proof.** For any open $U$ in $M$ and $x \in (V/L)(U) = \text{Hom}_M(U, V/L)$, we need to find an open cover $\{U_i\}$ of $U$ such that each $x|_{U_i} \in (V/L)(U_i)$ lifts to $V(U_i) = \text{Hom}_M(U_i, V)$. Equivalently (check!), this says that the projection $V \times_{V/L, x} U \to U$ admits holomorphic sections locally over $U$. But $V \to V/L$ is a covering map that is a local analytic isomorphism, so these properties are inherited by the base change $V \times_{V/L, x} U \to U$. Consequently, the existence of local holomorphic sections is clear. ■