Math 248B. Base change morphisms

1. Motivation

A basic operation with sheaf cohomology is pullback. For a continuous map of topological spaces $f : X' \to X$ and an abelian sheaf $\mathscr{F}$ on X with (topological) pullback $f^{-1}\mathscr{F}$ on X', there is a unique map of δ-functors

$$\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X', f^{-1}\mathscr{F}) \tag{1.1}$$

that is the usual pullback for $i = 0$.

Example 1.1. When $\mathscr{F}$ is a constant sheaf $\underline{A}$ on X associated to an abelian group A, so the same holds for $f^{-1}\mathscr{F}$ on X', then (for reasonable spaces) the general pullback construction in sheaf cohomology recovers the traditional pullback map

$$\mathrm{H}^i(X, A) \to \mathrm{H}^i(X', A)$$

as in algebraic topology.

Also, if f is a map of ringed spaces and $\mathscr{F}$ is an $\mathscr{O}_X$-module then we get the ringed-space pullback $f^*\mathscr{F} := \mathscr{O}_{X'} \otimes_{f^{-1}\mathscr{O}_X} f^{-1}\mathscr{F}$ and so a composite pullback map

$$\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X', f^{-1}\mathscr{F}) \to \mathrm{H}^i(X', f^*\mathscr{F}).$$

Remark 1.2. In the topological case (with reasonable spaces) and in the scheme case (with quasi-coherent sheaves) we compute sheaf cohomology in terms of Čech theory. Sometimes there are natural pullback maps in Čech theory, and it is natural to wonder if the above pullback maps in sheaf cohomology obtained via abstract δ-functor considerations are the same as what is computed more explicitly via Čech-theoretic methods. Let us state the desired compatibility in precise terms, and then we give a reference for its proof. (This compatibility is implicit in some proofs in Hartshorne's textbook "Algebraic Geometry", although it is never stated there.)

Consider a general map f of ringed spaces as above; this could be topological spaces equipped with the structure sheaf $\underline{\mathbf{Z}}$, or complex manifolds, or schemes, and so on. Let $\mathscr{F}' = f^*\mathscr{F}$ (so this is topological pullback in the topological case). Let $\mathfrak{U}$ be an open covering of X, and $\mathfrak{U}'$ an open covering of X' that refines the pullback open covering $f^{-1}(\mathfrak{U})$. Then using pullback of sections, we get a natural map $\mathrm{H}^i(\mathfrak{U}, \mathscr{F}) \to \mathrm{H}^i(\mathfrak{U}', \mathscr{F}')$. On the other hand, there are always natural maps from Čech theory to derived functor cohomology

$$\mathrm{H}^i(\mathfrak{U}, \mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F}), \;\; \mathrm{H}^i(\mathfrak{U}', \mathscr{F}') \to \mathrm{H}^i(X', \mathscr{F}')$$

(obtained from the Čech to derived functor spectral sequence, for example), and there are *isomorphisms* in special cases: constant sheaves on manifolds and "good covers" (contractible open sets with contractible higher overlaps), quasi-coherent sheaves on separated schemes and affine open covers, coherent analytic sheaves on complex manifolds and "Stein covers".

Consider the diagram

$$\begin{array}{ccc} \mathrm{H}^i(\mathfrak{U}, \mathscr{F}) & \longrightarrow & \mathrm{H}^i(\mathfrak{U}', \mathscr{F}') \\ \downarrow & & \downarrow \\ \mathrm{H}^i(X, \mathscr{F}) & \longrightarrow & \mathrm{H}^i(X', f^*\mathscr{F}) \end{array}$$

in which the vertical maps are the natural ones, the bottom is the abstract pullback, and the top is the concrete Čech-theoretic pullback. The compatibility we want to have is that this diagram commutes. This underlies many concrete calculations of pullback maps in sheaf cohomology. The commutativity of the diagram does hold in general, but the proof is not obvious. We refer the interested reader to EGA, 0_{III}, 12.1 (see especially the diagram (12.1.4.2)) for a proof.

In this handout, we want to discuss a relative version of the various pullback maps in sheaf cohomology. Consider a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{h}} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow[h]{} & M \end{array} \tag{1.2}$$

and an $\mathscr{O}_X$-module $\mathscr{F}$ with pullback $\mathscr{F}' := \widetilde{h}^*\mathscr{F}$ on X'. The three main cases of interest to us are the *topological case* (using structure sheaves $\underline{\mathbf{Z}}$ on everything, so $\mathscr{F}$ is an abelian sheaf on X and $\mathscr{F}'$ is its topological pullback on X'), the *scheme case* (using quasi-coherent sheaves subject to some finiteness hypotheses), and the *holomorphic case* with complex manifolds with f a submersion. In this latter case, we are most interested in $\mathscr{F}$ locally free of finite rank (i.e., associated to a vector bundle on X), so $\mathscr{F}'$ is the corresponding "vector bundle" pullback along $X' \to X$. In all three cases, the most important situations will have f proper and (1.2) cartesian (i.e., $X' = X \times_M M'$), but we do not make any such assumptions at the outset.

Remark 1.3. We use the ringed space language so that we can largely treat the topological and holomorphic cases at the same time, rather than write out everything twice. If $\mathscr{G}$ is an $\mathscr{O}_M$-module then its stalk $\mathscr{G}_m$ is an $\mathscr{O}_{M,m}$-module, so in the locally ringed case we can kill $\mathfrak{m}_m$ to get a vector space $\mathscr{G}(m) := \mathscr{G}_m/\mathfrak{m}_m\mathscr{G}_m$ over the residue field at m. In the topological setting (with structure sheaf $\underline{\mathbf{Z}}$) we only have stalks.

The higher direct images $\mathrm{R}^i f_*(\mathscr{F})$ are $\mathscr{O}_M$-modules, and in class we constructed a natural map

$$\mathrm{R}^i f_*(\mathscr{F})_m \to \mathrm{H}^i(X_m, \mathscr{F}|_{X_m}) \tag{1.3}$$

in the topological case and a $\mathbf{C}$-linear map

$$\mathrm{R}^i f_*(\mathscr{F})(m) \to \mathrm{H}^i(X_m, \mathscr{F}_m) \tag{1.4}$$

in the holomorphic case (where $\mathscr{F}_m$ on X_m denotes the ringed-space pullback of $\mathscr{F}$ along $X_m \to X$; this is not a stalk at m!).

Note that the construction (1.4) makes sense more generally in the setting of locally ringed spaces (including both schemes and complex manifolds), as a linear map over the residue field at m. Much as the higher direct image sheaves $\mathrm{R}^i f_*\mathscr{F}$ will be a useful device to "glue" fibral cohomologies $\mathrm{H}^i(X_m, \mathscr{F}_m)$ into a single global structure over M, for $m' \in M'$ we seek to "glue" the fibral pullback cohomology maps

$$\mathrm{H}^i(X_{h(m')}, \mathscr{F}_{h(m')}) \to \mathrm{H}^i(X'_{m'}, \mathscr{F}'_{m'})$$

into a map involving higher direct images relative to f and f'. More specifically, the aim of this handout is to *define* and analyze the functorial properties of a vast generalization: to any commutative square (1.2) we will associate canonical $\mathscr{O}_{M'}$-linear morphisms

$$h^*\mathrm{R}^i f_*(\mathscr{F}) \to \mathrm{R}^i f'_*(\mathscr{F}'). \tag{1.5}$$

These will be called *base change* morphisms when we are in the topological, scheme, or holomorphic cases and (1.2) is cartesian.

Example 1.4. In the special case that h is the inclusion of a point and the left side of (1.2) is the m-fiber of f in the topological case (resp. scheme or holomorphic cases), the map (1.5) will recover (1.3) (resp. (1.4)). The generality of (1.5), allowing rather arbitrary h, is very useful.

In §12 of Chapter III of Hartshorne (as well as the end of Vakil's notes) there is a discussion of (1.5) for certain cartesian diagrams of sheaves (with suitable $\mathscr{F}$ and proper f). The main point of that discussion is the understanding of fibral criteria of Grothendieck which imply that both sides of (1.5) are locally free of finite rank and the map (1.5) is an isomorphism. These are called "cohomology and base change" theorems. They provide a precise sense in which fibral cohomologies $\mathrm{H}^i(X_m, \mathscr{F}_m)$ can be said to "vary nicely in m" (under suitable fibral hypotheses!).

Remark 1.5. A confusing aspect of the discussion of cohomology and base change in Hartshorne's textbook is that the construction and functorial properties of (1.5) are never systematically discussed there, yet they pervade the (omitted) rigorous details of many of the proofs. Hence, in effect the present discussion logically precedes any treatment of the "cohomology and base change" in the scheme or holomorphic cases.

Strictly speaking, Hartshorne only discusses the cohomology and base change theorems for projective f; the general case is in Vakil's notes (and also in EGA III). There are analogous results in the holomorphic setting, due to Kodaira–Spencer and Grauert. In class we discussed some reasons why these results are useful in the analytic setting, and in our later study of elliptic curves over schemes we will find them to be equally useful. Briefly, whenever an argument with elliptic curves in the classical setting over a field uses Riemann–Roch, the proof of a corresponding result in the relative setting in either the holomorphic or scheme cases will use the theorems on cohomology and base change to reduce difficult problems in the relative setting to solved problems in the classical setting on fibers.

2. Constructions and examples

To construct (1.5), first observe that for any open U in M with preimage $U' = h^{-1}(U)$ in M', the preimage of $X_U := f^{-1}(U)$ in X' is $X'_{U'} := {f'}^{-1}(U')$ due to the commutative of (1.2). Hence, there is a natural pullback map

$$\mathrm{H}^i(X_U, \mathscr{F}) \to \mathrm{H}^i(X'_{U'}, \mathscr{F}').$$

But in view of the description of higher direct images as sheafifications, there is a natural map

$$\mathrm{H}^i(X'_{U'}, \mathscr{F}') = \mathrm{H}^i({f'}^{-1}(U'), \mathscr{F}') \to (\mathrm{R}^i f'_* \mathscr{F}')(U')$$

whose target is $\Gamma(h^{-1}(U), \mathrm{R}^i f'_* \mathscr{F}') = \Gamma(U, h_*(\mathrm{R}^i f'_* \mathscr{F}'))$. Summarizing, for open U in M we have constructed natural maps

$$\mathrm{H}^i(f^{-1}(U), \mathscr{F}) \to \Gamma(U, h_*(\mathrm{R}^i f'_* \mathscr{F}')) \tag{2.1}$$

which are easily checked to respect restriction in U.

Since $h_*(\mathrm{R}^i f'_* \mathscr{F}')$ is a sheaf whereas $U \mapsto \mathrm{H}^i(f^{-1}(U), \mathscr{F})$ is merely a presheaf whose sheafification is $\mathrm{R}^i f'_* \mathscr{F}$, by the universal property of sheafification we obtain from the maps (2.1) a natural map

$$\mathrm{R}^i f_* \mathscr{F} \to h_*(\mathrm{R}^i f'_* \mathscr{F}')$$

of sheaves on M. By the construction this is seen to be $\mathscr{O}_M$-linear, so by the adjointness of h_* and h^* we obtain an $\mathscr{O}_{M'}$-linear map as in (1.5). This completes the construction.

Example 2.1. Suppose that (1.2) is a diagram of topological spaces or locally ringed spaces (such as complex manifolds or schemes) with h the inclusion of a point $m' \in M'$. Then $h^*\mathscr{G}$ is the stalk $\mathscr{G}_{m'}$ in the topological case and is $\mathscr{G}(m) := \mathscr{G}_m / \mathfrak{m}_m \mathscr{G}_m$ in the locally ringed space case. In this way, one sees upon unraveling the construction that (1.5) recovers (1.3) in the topological case and (1.4) in the locally ringed case.

Example 2.2. If h is an open immersion and (1.2) is cartesian (i.e., X' is the preimage of an open $M' \subset M$ under the map $f : X \to M$), then (1.5) is an isomorphism expressing the fact that the formation of higher direct image sheaves is local on the base.

Example 2.3. In general, what is (1.5) saying at the level of stalks? For $m' \in M'$, the map on m'-stalks has the form

$$\mathscr{O}_{M',m'} \otimes_{\mathscr{O}_{M,m}} \varinjlim \mathrm{H}^i(X_U, \mathscr{F}) \to \varinjlim \mathrm{H}^i(X'_{U'}, \mathscr{F}')$$

where U varies through opens in M around $h(m')$ and U' varies through opens in M' around m'. A special class of such U' are the opens $h^{-1}(U)$, and in fact the m'-stalk map is induced by the natural pullback maps

$$\mathrm{H}^i(X_U, \mathscr{F}) \to \mathrm{H}^i(X'_{h^{-1}(U)}, \mathscr{F}') \tag{2.2}$$

linear over $h^\sharp : \mathscr{O}_M(U) \to (h_* \mathscr{O}_{M'})(U) = \mathscr{O}_{M'}(h^{-1}(U))$, as is checked by reviewing how (1.5) was constructed. This description is a reason that we regard (1.5) as a relativization of ordinary pullback maps in sheaf cohomology.

3. Functorial properties

We conclude by discussing how (1.5) behaves with respect to δ-functoriality in $\mathscr{F}$ and with respect to compositions in the horizontal and vertical directions: loosely speaking, functional behavior with respect to composition in h and in f.

For the δ-functoriality, suppose that $M' \to M$ and $X' \to X$ are flat maps of ringed spaces, such as submersions of manifolds, flat maps of schemes, or maps of topological spaces equipped with the structure sheaf $\underline{\mathbf{Z}}$. In such cases $\mathscr{F}'$ is an exact functor in $\mathscr{F}$, so the target of (1.5) is a δ-functor in $\mathscr{F}$. Likewise, h^* is exact, so the source of (1.5) is δ-functorial in $\mathscr{F}$. This leads to:

Proposition 3.1. *When $M' \to M$ and $X' \to X$ are flat then* (1.5) *is a morphism of δ-functors in $\mathscr{F}$; that is, it respects the δ-functorial structure of both sides as we vary $\mathscr{F}$ in a short exact sequence of $\mathscr{O}_X$-modules.*

Proof. It suffices to check the δ-functorial compatibility after passing to stalks. In view of the description on stalks given in Example 2.3, it suffices to prove that (2.2) is a morphism of δ-functors for each open $U \subset M$. Renaming U as M (as we may clearly do), we are reduced to checking that the composite map

$$\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X', f^{-1}\mathscr{F}) \to \mathrm{H}^i(X', f^*\mathscr{F})$$

is a morphism of δ-functors. This holds for the first step by the very definition of topological pullback in sheaf cohomology, and for the second step (which is defined to be the cohomology functor applied to the natural map of sheaves $f^{-1}\mathscr{F} \to f^*\mathscr{F}$) are use that cohomology is defined as a *δ-functor*. ∎

The compatibility with respect to composition in h works out as follows. Consider a concatenated pair of commutative squares

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ M'' & \xrightarrow[h']{} & M' & \xrightarrow[h]{} & M \end{array}$$

and let $h'' : M'' \to M$ denote $h \circ h'$. We then get the pullbacks $\mathscr{F}'$ on X' and $\mathscr{F}''$ on X'', so $\mathscr{F}''$ is also naturally identified with the pullback of $\mathscr{F}'$ along $X'' \to X'$. Thus, there are natural morphisms

$$\theta : h^*\mathrm{R}^i f_*(\mathscr{F}) \to \mathrm{R}^i f'_*(\mathscr{F}'), \quad \theta' : h'^*\mathrm{R}^i f'_*(\mathscr{F}') \to \mathrm{R}^i f''_*(\mathscr{F}''), \quad \theta'' : h''^*\mathrm{R}^i f_*(\mathscr{F}) \to \mathrm{R}^i f''_*(\mathscr{F}'').$$

The consistency among these morphisms is given by:

Proposition 3.2. *The composition $\theta' \circ h'^*(\theta)$ is equal to θ''.*

Proof. Again passing to stalks and arguing as in the preceding proof, we are reduced to the general functoriality of sheaf cohomology pullback. That is, for a pair of maps of topological spaces $h : X' \to X$ and $h' : X'' \to X'$ with composite $h'' = h \circ h'$ and any abelian sheaf $\mathscr{F}$ on X, we claim that the composition of pullback maps

$$\mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X', h^{-1}\mathscr{F}) \to \mathrm{H}^i(X'', h'^{-1}(h^{-1}\mathscr{F})) = \mathrm{H}^i(X'', h''^{-1}\mathscr{F})$$

is the pullback along h''. (This is a fact one should have really verified at the beginning of the entire story, when first introducing sheaf cohomology pullback and contemplating its functorial behavior.) In view of the δ-functoriality of everything in sight (as we vary $\mathscr{F}$), this problem reduces to the case $i = 0$, which is easily verified. ∎

The case of behavior in the "vertical" direction is somewhat more subtle to even formulate, and for our purposes will not be nearly as important as the "horizontal" compatibility in the preceding proposition.

Consider a commutative diagram of ringed spaces

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{h'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow[h]{} & M \end{array}$$

and an $\mathscr{O}_Y$-module $\mathscr{G}$. Thus, (1.5) provides a natural morphism

$$h^*\mathrm{R}^i(f\circ g)_*(\mathscr{G}) \to \mathrm{R}^i(f'\circ g')_*(\mathscr{G}')$$

where $\mathscr{G}'$ is the ringed-space pullback of $\mathscr{G}$ along $Y' \to Y$. There are Leray spectral sequences

$$\mathrm{R}^p f_*\mathrm{R}^q g_*(\mathscr{G}) \Rightarrow \mathrm{R}^{p+q}(f\circ g)_*(\mathscr{G}),\ \ \mathrm{R}^p g_*\mathrm{R}^q g'_*(\mathscr{G}) \Rightarrow \mathrm{R}^{p+q}(f'\circ g')_*(\mathscr{G}),$$

and when h and h' are flat (so h^* and h'^* are exact and hence commute with the formation of kernels and images, as arise in the formation of spectral sequences) it makes sense to ask if these spectral sequences are compatible with (1.5) relative to the smaller commutative squares. This is the content of:

Proposition 3.3. *When h and h' are flat, the natural maps*

$$h^*(\mathrm{R}^p f_*\mathrm{R}^q g_*(\mathscr{G})) \to \mathrm{R}^p f'_* h'^*(\mathrm{R}^q g_*\mathscr{G}) \to \mathrm{R}^p f'_*\mathrm{R}^q g'_*(\mathscr{G}')$$

and

$$h^*\mathrm{R}^n(f\circ g)_*(\mathscr{G}) \to \mathrm{R}^n(f'\circ g')_*(\mathscr{G}')$$

are compatible with the formation of the associated spectral sequences.

Proof. By passing to stalks as in the two preceding proofs, this again reduces to an assertion about pullback in topological sheaf cohomology: it commutes with the formation of Leray spectral sequences such as

$$\mathrm{H}^p(X, \mathrm{R}^q g_*\mathscr{G}) \Rightarrow \mathrm{H}^{p+q}(Y, \mathscr{G})$$

and the analogue for $X' \to Y' \to M'$. This in turn is seen by going back to how spectral sequences are constructed in terms of Cartan–Eilenberg resolutions and how sheaf cohomology pullback may be computed in terms of injective resolutions (due to the characterization of the latter in terms of δ-functoriality). We leave the details to the interested reader. ∎