Math 248A. Homework 9

1. Let p be a prime. Prove that the only elements of $\mathbf{Q}_p^\times$ that admit eth roots for all e relatively prime to p are the elements in $1+p\mathbf{Z}_p$, and deduce that all automorphisms of $\mathbf{Q}_p$ are continuous and hence equal the identity. Also prove that $\mathbf{Q}_p \not\simeq \mathbf{Q}_{p'}$ as abstract fields for any $p' \neq p$.

2. Let F be the fraction field of a Dedekind domain A. Let $\mathfrak{m}$ be a maximal ideal of A and let $F_{\mathfrak{m}}$ be the corresponding completion of F, with valuation ring $A_{\mathfrak{m}}^\wedge$.

(i) If I and J are two fractional ideals of F, prove that $I=J$ in A if and only if $IA_{\mathfrak{m}}^\wedge = JA_{\mathfrak{m}}^\wedge$ inside of $F_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A. (Hint: First reduce to the case when A is a discrete valuation ring.)

(ii) Let F'/F be a finite separable extension, and let A' be the integral closure of A in F' (so A' is a finite A-module). Let $\mathfrak{m}_i'$ ($1 \leq i \leq g$) be the maximal ideals of A' over $\mathfrak{m}$, and let $F'_{\mathfrak{m}_i'}$ be the corresponding completion of F' (so it is a finite separable extension of $F_{\mathfrak{m}}$, as is shown in the study of completions in the handout). Let $A'^\wedge_{\mathfrak{m}_i'}$ be the valuation ring of $F'_{\mathfrak{m}_i'}$.

Prove that $A'^\wedge_{\mathfrak{m}_i'}$ is the integral closure of $A_{\mathfrak{m}}^\wedge$ in $F'_{\mathfrak{m}_i'}$, and that the nonzero product of local discriminants $\prod_i \mathfrak{d}_{A'^\wedge_{\mathfrak{m}_i'}/A^\wedge_{\mathfrak{m}}}$ in $A_{\mathfrak{m}}^\wedge$ is equal to $\mathfrak{d}_{A'/A}A_{\mathfrak{m}}^\wedge$; in this sense, formation of the discriminant is "compatible" with completion. (Hint: reduce to the case when A is a discrete valuation ring.)

(iii) If $f \in F[X]$ is a monic separable polynomial with positive degree and F'/F is a splitting field, prove that a maximal ideal $\mathfrak{m}$ of A is unramified in F' if and only if the splitting field for f over $F_{\mathfrak{m}}$ is unramified.

3. Let K be a field that is complete with respect to a non-trivial non-archimedean absolute value $|\cdot|$.

(i) For $a_0, a_1, \dots \in K$, define $r = 1/\limsup |a_n|^{1/n} \in [0,\infty]$. Prove that $\sum a_n x^n$ converges for $x \in K$ if $|x| < r$ and does not converge if $|x| > r$. Prove that if it converges for one x_0 with $|x_0| = r$ then it converges for all x with $|x| = r$. Also prove that if $r > 0$ then $\sum a_n x^n = 0$ for all x near 0 only if $a_n = 0$ for all n.

(ii) Assume that K has characteristic 0 and that its residue field k has characteristic p. Replace $|\cdot|$ with a suitable power so that $|p| = 1/p$. Prove that the power series $\log_p(1+x) = \sum_{n\geq 1}(-1)^{n+1}x^n/n$ converges if and only if $|x| < 1$, and that $\log_p((1+x)(1+y)) = \log_p(1+x+y+xy)$ is equal to $\log_p(1+x)+\log_p(1+y)$ for all $|x|, |y| < 1$. (Be rigorous!) Also prove that $|\log_p(1+x)| = |x|$ if $|x| < p^{-1/(p-1)}$.

(iii) Prove that $\mathrm{ord}_p(n!) = (n - S_n)/(p-1)$ for a positive integer n, where S_n is the sum of the digits in the ordinary base-p decimal expansion of n. Conclude that for K as in (ii), the formal power series $\exp_p(x) = \sum x^n/n!$ converges if and only if $|x| < p^{-1/(p-1)}$, and that $|\exp_p(x) - 1| = |x|$ for all such x. Prove that $\exp_p(x+y) = \exp_p(x)\exp_p(y)$ for $|x|, |y| < p^{-1/(p-1)}$.

(iv) Prove that open disc $|t-1| < r$ in K (as in (ii)) is a subgroup of $K^\times$ whenever $r < 1$, and show that $t \mapsto \log_p(t)$ and $x \mapsto \exp_p(x)$ define inverse isomorphism between the open disc $|t-1| < p^{-1/(p-1)}$ and the open disc $|x| < p^{-1/(p-1)}$.

(v) Taking $K = \mathbf{Q}_p$, prove that $\log_p$ maps the multiplicative group $1+p\mathbf{Z}_p$ homeomorphically onto the additive group $p\mathbf{Z}_p$ if $p > 2$ and that it maps $1+4\mathbf{Z}_2$ homeomorphically onto the additive group $4\mathbf{Z}_2$ if $p = 2$. Conclude that all elements of $1+8\mathbf{Z}_2$ have square roots in $\mathbf{Q}_2$, and as an application describe all quadratic extensions of $\mathbf{Q}_p$ for all primes p (the case $p=2$ requires separate treatment).

Deduce that the imaginary quadratic field $\mathbf{Q}(\sqrt{-1})$ and the real quadratic field $\mathbf{Q}(\sqrt{7})$ that are ramified at 2 have $\mathbf{Q}_2$-isomorphic completions at their unique primes over 2.

4. (i) Prove that if K is a non-archimedean local field with residue characteristic p and $n \in \mathbf{Z}$ is nonzero, then $(K^\times)^n$ is open in $K^\times$ with finite index if $p \nmid n$ (hint: Hensel's lemma), and prove the same for $\mathrm{char}(K) = 0$ and *any* $n \in \mathbf{Z} - \{0\}$ by using the p-adic logarithm. Deduce that every subgroup of finite index in $K^\times$ is open if $\mathrm{char}(K) = 0$.

(ii) If K is a local field with positive characteristic p, then use the description of K as $k((t))$ to show that $(K^\times)^p$ is closed but not open in $K^\times$. Also prove the existence of a finite-index subgroup of $K^\times$ that is not open. (Hint: Prove that $K^\times/(K^\times)^p$ is an $\mathbf{F}_p$-vector space with uncountable dimension.)

5. (optional) Let K be a (non-archimedean) local field with normalized absolute value $\|\cdot\|_K$. Let $\mathrm{d}x$ denote a Haar measure on K. Prove that $\mathrm{d}x/\|x\|_K$ on the open subset $K^\times$ is a Haar measure for $K^\times$. (Hint: $x \mapsto \|x\|_K$ is locally constant on $K^\times$)

6. (optional) Let K be a global function field; that is, a finitely generated extension of some $\mathbf{F}_p$ with transcendence degree 1. Let the finite field k denote the algebraic closure of $\mathbf{F}_p$ in K. (This is the *constant field* of K.) Since k is perfect, there exists a *separating transcendence basis*: an element $x \in K^\times$ transcendental over k such that K is finite separable over $k(x)$.

(i) Prove that the set M_K of topological equivalence classes of non-trivial absolute values on K is countable, and that for each such equivalence class v the associated valuation ring A_v is a discrete valuation ring with fraction field K and residue field $\kappa(v)$ that is finite. (Hint: Use an extension structure $K/k(x)$ to reduce to the case $K = k(x)$.) For each v, we write $|\cdot|_v$ to denote the unique representative for v whose value group is $q_v^{\mathbf{Z}}$, where $q_v = \#\kappa(v)$. Prove that for each $f \in K^\times$ we have $|f|_v = 1$ for all but finitely many $v \in M_K$, and prove that $|f|_v = 1$ for *all* $v \in M_K$ if and only if $f \in k^\times$; in other words, just like for number fields, $|f|_v = 1$ for all v if and only if f is a root of unity! (Hint: study a minimal polynomial for f over $k(x)$)

(ii) Prove the *product formula*: $\prod_v |f|_v = 1$ for all $f \in K^\times$. (Hint: Use an extension structure $K/k(x)$ to reduce to the case $K = k(x)$, imitating the method of reduction to $\mathbf{Q}$ in the number field case.)

(iii) Let S be a finite non-empty set in M_K. The *Riemann-Roch theorem* over k, applied to K/k, ensures that there exists a separating transcendence basis x for K/k such that $K/k(x)$ is not only finite and separable but even has S as exactly the set of places over the infinite place on $k(x)$.

The ring $\mathscr{O}_{K,S}$ of *S-integers* in K is the set of $f \in K$ such that $f \in A_v$ for all $v \notin S$. Use an x as above to prove that $\mathscr{O}_{K,S}$ is a Dedekind domain finitely generated as an k-algebra, and that its maximal ideals are in one-to-one correspondence with the places on K outside of S. Moreover, if a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_{K,S}$ corresponds to a place $v \notin S$, then prove that the algebraic localization $(\mathscr{O}_{K,S})_{\mathfrak{m}}$ inside of K is equal to the valuation ring A_v.

(iv) The *adele ring* $\mathbf{A}_K \subseteq \prod_v K_v$ is the directed union of subrings $\mathbf{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathscr{O}_v$. Using the evident locally compact Hausdorff topological ring structure on the $\mathbf{A}_{K,S}$'s, explain how to give $\mathbf{A}_K$ a structure of locally compact Hausdorff topological ring. Also prove that in general $\mathbf{A}_K^\times$ with its subspace topology is not open in $\mathbf{A}_K$ and is not a topological group. (hint: For each non-archimedean place v, let $x_v \in \mathbf{A}_K$ be the adele with v-coordinate π_v and v'-coordinate 1 for all $v' \neq v$. Prove $x_v \in \mathbf{A}_K^\times$ with $\{x_v\}$ converging to 1 in $\mathbf{A}_K$ (for any choice of enumeration of M_K) but $\{x_v^{-1}\}$ has no limit in $\mathbf{A}_K$.)

7. (optional) Let K'/K be a finite separable extension of global fields. (Separability can be dropped, but this requires more commutative algebra than we have developed.)

(i) Define a natural continuous map $\mathbf{A}_K \to \mathbf{A}_{K'}$ over $K \to K'$ that is a homeomorphism onto a closed subring. (Hint: Study places of K' over a fixed place of K.)

(ii) Prove that the natural map of K'-algebras $K' \otimes_K \mathbf{A}_K \to \mathbf{A}_{K'}$ is a topological isomorphism, where the left side is given the product topology upon using any K-basis of K'. (Why does this latter topology not depend on the choice of K-basis of K'?)

(iii) Use (ii) to prove that K' is discrete in $\mathbf{A}_{K'}$ with compact quotient for any global field K' by reduction to the special cases $K' = \mathbf{Q}$ and $K' = k(x)$ for a finite field k.

(iv) If R is a topological ring, show that giving $R^\times$ the subspace topology from $R \times R$ via the identification $x \mapsto (x, x^{-1})$ onto the subset $\{(x,y) \in R^2 \mid xy = 1\}$ gives $R^\times$ a structure of topological group that is functorial in R. Applying this with $R = \mathbf{A}_K$ to give $\mathbf{A}_K^\times$ a structure of topological group (this is the *only* topology ever put on $\mathbf{A}_K^\times$), use discreteness of K in $\mathbf{A}_K$ to infer discreteness of $K^\times$ in $\mathbf{A}_K^\times$ and use openness of $\mathbf{A}_{K,S}$ in $\mathbf{A}_K$ to infer openness of $\mathbf{A}_{K,S}^\times$ in $\mathbf{A}_K^\times$. Show that each subset $\mathbf{A}_{K,S}^\times \subseteq \mathbf{A}_K^\times$ is given a product topology, and use this to describe a base of opens around the identity in $\mathbf{A}_K^\times$. The topological group $\mathbf{A}_K^\times$ is the *idele group* of K.