

MATH 248A. HOMEWORK 11 (FOR HOLIDAY FUN!)

1. Let F be a field endowed with a discretely-valued non-archimedean place v , and let F_v be the completion. Let F_s and $F_{v,s}$ denote separable closures of F and F_v respectively. Let \bar{v} be a place on F_s lifting v . In Exercise 4 of Homework 10 you showed that there exists an embedding $F_s \rightarrow F_{v,s}$ over $F \rightarrow F_v$ that induces \bar{v} , and that this embedding is unique up to the action of the closed subgroup $D(\bar{v}|v) \subseteq \text{Gal}(F_s/F)$. Use this embedding to construct a continuous map of topological groups $\text{Gal}(F_{v,s}/F_v) \rightarrow D(\bar{v}|v)$ that is unique up to inner automorphism of $D(\bar{v}|v)$, and use Krasner's lemma to prove that this continuous map is an isomorphism of topological groups.

In particular, the map $\text{Gal}(F_{v,s}/F_v) \rightarrow \text{Gal}(F_s/F)$ that is canonical up to inner automorphism of $\text{Gal}(F_s/F)$ is injective, and in fact is a topological isomorphism onto a closed subgroup of $\text{Gal}(F_s/F)$.

2. Let $G = \text{Gal}(\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p)$ for a prime p and $n \geq 1$. Let G_i denote the i th ramification group (so G_0 is the inertia subgroup, and $G_i = 1$ for large i). For $i \geq 0$, use the definition of the higher ramification groups to prove that $G_i = \text{Gal}(\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p(\zeta_{p^j}))$ for $p^{j-1} \leq i \leq p^j - 1$ when $i < p^{n-1}$, and $G_i = 1$ for $i \geq p^{n-1}$.

3. Let K be a field. A \mathbf{Z}_p -extension of K is an abelian extension K'/K such that $\text{Gal}(K'/K) \simeq \mathbf{Z}_p$ as topological groups.

(i) Let K'/K be any \mathbf{Z}_p -extension. Prove that for each $n \geq 1$ there exists a unique intermediate extension with degree p^n over K , and that this exhausts the set of *all* intermediate subfields not equal to K and K' . (Hint: \mathbf{Z} is topologically dense in \mathbf{Z}_p .)

(ii) Let K be a number field and let $K' = K(\mu_{p^\infty})$ be the extension generated by all p -power roots of unity. Prove that the natural map $\text{Gal}(K'/K) \rightarrow \text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^\times$ is an isomorphism onto a closed subgroup with finite index, and so conclude that it is an isomorphism onto an open subgroup. Deduce via the p -adic logarithm and the absence of non-trivial torsion in \mathbf{Z}_p that $\text{Gal}(K'/K)$ has a *unique* quotient topologically isomorphic to \mathbf{Z}_p , and it corresponds to a subextension K_∞ that is a \mathbf{Z}_p -extension of K and over which K' has finite degree. We call K_∞ the *cyclotomic \mathbf{Z}_p -extension* of K .

(iii) Let K be a local field with residue characteristic not equal to p . Explain why its cyclotomic \mathbf{Z}_p -extension is unramified, and use Galois theory for finite fields to prove that this is the unique unramified \mathbf{Z}_p -extension of K . Use (i) above and Exercise 4(ii) below to prove that any \mathbf{Z}_p -extension *must* be unramified. (Hint: Show that the image of inertia must be tame, and contrast how Frobenius lifts act on tame inertia through conjugation as in Exercise 4(ii) and how conjugation behaves in the abelian group \mathbf{Z}_p .)

(iv) Let K'/K be a \mathbf{Z}_p -extension of a number field. Using the absence of non-trivial torsion in \mathbf{Z}_p , prove that if v is a real place on K then its extensions to K' are real. (In particular, if K is totally real then all embeddings of K' into \mathbf{C} are land in \mathbf{R} .) Use (iii) to prove that if v is a non-archimedean place of K with residue characteristic not equal to p then v is *unramified* in K' . Hence, K'/K is unramified away from the finite set of non-archimedean places of K with residue characteristic p .

4. Let F be a field that is complete with respect to a discretely-valued non-archimedean place v , and let F_s/F denote a separable closure. Let F_{un}/F denote the maximal unramified subextension (that is, the directed union of unramified finite subextensions), and let F_t/F denote the maximal tamely ramified subextension (that is, the directed union of tamely ramified finite subextensions). Endow these fields with their unique place extending that on F . The *tame quotient* I_t of the inertia group $I = \text{Gal}(F_s/F_{\text{un}})$ is $\text{Gal}(F_t/F_{\text{un}})$.

(i) Explain why F_{un} is discretely-valued with residue field $k(v)'$ that is a separable closure of the residue field $k(v)$ at v on F . Prove that it is not complete if $[k(v)_{\text{sep}} : k(v)]$ is infinite.

(ii) Prove that every finite subextension of F_t/F_{un} is totally tamely ramified, and use Kummer theory to construct a canonical isomorphism of topological groups $I_t \simeq \varprojlim_e \mu_e(k(v)')$ where the inverse limit is taken over $e \geq 1$ not divisible by $\text{char}(k(v))$ and the transition maps in the inverse limit are the surjective (e'/e) th-power maps $\mu_{e'} \rightarrow \mu_e$ for $e|e'$. By using the short exact sequence

$$1 \rightarrow I_t \rightarrow \text{Gal}(F_t/F) \rightarrow \text{Gal}(k(v)'/k(v)) \rightarrow 1$$

with I_t abelian, explain how there results a natural action of $\text{Gal}(k(v)'/k(v))$ on I_t through conjugation in $\text{Gal}(F_t/F)$, and describe this action in terms of the action of $\text{Gal}(k(v)'/k(v))$ on roots of unity in $k(v)'$ with order not divisible by $\text{char}(k(v))$.

(iii) By choosing suitably compatible primitive roots of unity, construct a non-canonical isomorphism of topological groups $I_t \simeq \prod' \mathbf{Z}_\ell$, where \prod' means that ℓ ranges over primes not divisible by $\text{char}(k(v))$.