Math 248A. Transitivity of discriminants

Let $F''/F'/F$ be a tower of finite separable extensions of fields (so F''/F is finite and separable too), and let A be a Dedekind domain with fraction field F. Let A' and A'' be the respective integral closures of A in F' and F'', so each is Dedekind (and module-finite over A) with A'' also the integral closure of A' in F''. We therefore have three (nonzero) discriminant ideals: $\mathfrak{d}_{A''/A}$ and $\mathfrak{d}_{A'/A}$ are nonzero ideals in A and $\mathfrak{d}_{A''/A'}$ is a nonzero ideal in A'. It is natural to ask if $\mathfrak{d}_{A''/A}$ may be computed in terms of $\mathfrak{d}_{A''/A'}$ and $\mathfrak{d}_{A'/A}$. There is indeed a relation among these discriminants, via a formula called "transitivity of the discriminant". It goes as follows:

$$\mathfrak{d}_{A''/A} = \mathfrak{d}_{A'/A}^{[F'':F']} \mathrm{N}_{A'/A}(\mathfrak{d}_{A''/A'})$$

where $\mathrm{N}_{A'/A}$ is the "ideal norm" operation (discussed in class) that is a map from the group of nonzero fractional ideals of A' to the group of nonzero fractional ideals of A.

The purpose of this handout is to prove the transitivity formula for discriminants. Note the interesting consequence that the primes of A that ramify in A'' are precisely those that ramify in A' or that lie beneath a prime of A' that ramifies in A''; this assertion will become very easy to see by a different method later on when we study completions of fields. It should also be noted that a deeper insight into the transitivity of discriminants may be obtained via the theory of the so-called *different ideal* $\mathfrak{D}_{A'/A}$. (In contrast with the discriminant, the theory of the different is specific to the context of Dedekind domains.) The different $\mathfrak{D}_{A'/A}$ is a canonical ideal in A' whose norm down to A is $\mathfrak{d}_{A'/A}$, and there is a transitivity formula for the different that is somewhat simpler in appearance than the one for the discriminant and that in fact implies the transitvity formula for the discriminant. We refer the reader to Serre's book "Local Fields" for a lucid treatment of the different and its relationship with the discriminant. One "advantage" to our method of proof of transitivity of the discriminant is that (with stronger technique in commutative algebra) it carries over almost *verbatim* to work in very general situations of "finite locally free" ring extensions for which one has a theory of the discriminant ideal but no theory of the different ideal. That is, transitivity of discriminants holds in situations that go far beyond that of Dedekind domains (and it is extremely useful; one spectacular application is to be found in Tate's fundamental work on p-divisible groups). However, as with the development of the discriminant in the earlier handout, our "more general" technique requires using a heavy amount of linear algebra.

Let us now turn to the task of proving the proposed transitivity formula. This formula is an equality among ideals of A, so to verify it we need only check equality after localization at each maximal ideal of A. Since all ingredients (discriminants and ideal-norms) are compatible with localization at any multiplicative set of the base (not containing 0), by localizing throughout at $S = A - \mathfrak{m}$ we may assume that A is a discrete valuation ring. Hence, A' is semi-local, so both A and A' are PID's and thus A' is a free A-module and A'' is a free A'-module (both with finite rank). It follows that A'' is a free A-module. The transitivity formula may therefore be rewritten as:

$$\mathrm{disc}(A''/A) \stackrel{?}{=} \mathrm{disc}(A'/A)^{[F'':F']}\mathrm{N}_{A'/A}(\mathrm{disc}(A''/A')).$$

Since $\mathrm{disc}(A''/A')$ is a principal ideal, the norm of this ideal down to A may be computed as the principal ideal of A generated by the norm of any of the habitual determinants that provide a generator of $\mathrm{disc}(A''/A')$.

The transitivity of traces for field extensions (as in the earlier handout on norms and traces) implies a corresponding transitivity result for traces at the level of our Dedekind domains, and this implies that the composite A-bilinear pairing

$$A'' \otimes_A A'' \to A'' \otimes_{A'} A'' \xrightarrow{\mathrm{Tr}_{A''/A'}} A' \xrightarrow{\mathrm{Tr}_{A'/A}} A$$

(whose second and third steps are the bilinear trace pairings for A'' over A' and for A' over A) is the bilinear trace pairing for A'' over A'. Thus, we may now turn our attention to the following more general problem in linear algebra. Let A be a ring and let $A \to B$ be a ring map with B finite and free as an A-module. Let M be a finite free B-module with rank $r > 0$ (so M is also finite and free as an A-module), and let

$$\langle \cdot, \cdot \rangle_B : M \otimes_B M \to B$$

be a symmetric B-bilinear form. Define the symmetric A-bilinear form

$$\langle\cdot,\cdot\rangle_A : M \otimes_A M \to M \otimes_B M \overset{\langle\cdot,\cdot\rangle_B}{\to} B \overset{\mathrm{Tr}_{B/A}}{\to} A$$

to be (roughly) the composite $\mathrm{Tr}_{B/A} \circ \langle\cdot,\cdot\rangle_B$. It makes sense to form the discriminants $\mathrm{disc}(\langle\cdot,\cdot\rangle_B)$ and $\mathrm{disc}(\langle\cdot,\cdot\rangle_A)$ as principal ideals of B and A generated by the habitual determinants in B and A that are well-defined up to unit square multiple. In this generality, we claim

$$\mathrm{disc}(\langle\cdot,\cdot\rangle_A) = \mathrm{disc}(B/A)^r \mathrm{N}_{B/A}(\mathrm{disc}(\langle\cdot,\cdot\rangle_B))$$

as ideals of A, where the norm on the right is understood to mean the principal ideal generated by the ring-theoretic norm of any of the habitual determinants that provide a generator for the discriminant ideal in B attached to $\langle\cdot,\cdot\rangle_B$.

Let $\psi_B : M \to \mathrm{Hom}_B(M,B)$ be the map $m \mapsto \langle m,\cdot\rangle_B = \langle\cdot,m\rangle_B$ between finite free B-modules with the same rank r. The corresponding map $\psi_A : M \to \mathrm{Hom}_A(M,A)$ between finite free A-modules with the same rank may be computed as the composite

$$\psi_A : M \overset{\psi_B}{\to} \mathrm{Hom}_B(M,B) \to \mathrm{Hom}_A(M,B) \overset{\mathrm{Tr}_{M,B/A}}{\to} \mathrm{Hom}_A(M,A).$$

The determinants $\det(\psi_B)$ and $\det(\psi_A)$ are well-defined in B and A up to unit multiple (by choosing B-bases and A-bases on the source and target for ψ_B and ψ_A respectively). The elements $\det(\psi_B)$ and $\det(\psi_A)$ are unit multiples of generators for the discriminants of $\langle\cdot,\cdot\rangle_B$ and $\langle\cdot,\cdot\rangle_A$. (If we use bases and dual bases then we obtain the habitual determinants that define discriminant ideals.)

Ignoring units in A and B, we are reduced to showing that $\det_A(\psi_A)$ and $\mathrm{disc}(B/A)^r \mathrm{N}_{B/A}(\det_B(\psi_B))$ are off by $A^\times$-multiple (where $\mathrm{disc}(B/A)$ denotes any of the usual determinants for the A-bilinear trace pairing from B to A). If r' denotes the A-rank of B then for any finite free B-module N with rank r (so the underlying A-module N_A is finite and free with rank rr'), we have a canonical isomorphism

$$\wedge_A^{rr'}(N_A) \simeq \wedge_A^{r'}(\wedge_B^r N),$$

so by the *definition* of $\mathrm{N}_{B/A}$ as a determinant we have $\mathrm{N}_{B/A}(\det_B \psi_B) = \det_A(\psi_B)$. By multiplicativity of $\det_A$ for maps between finite free A-modules of the same finite rank, coupled with the above description of ψ_A as a 3-step composite involving ψ_B (considered as an A-linear map), the forgetful map $\phi : \mathrm{Hom}_B(M,B) \to \mathrm{Hom}_A(M,B)$, and the trace map $\mathrm{Tr}_{M,B/A} : \mathrm{Hom}_A(M,B) \to \mathrm{Hom}_A(M,A)$, it suffices to prove

$$\det_A(\mathrm{Tr}_{M,B/A} \circ \phi) \overset{?}{=} \mathrm{disc}(B/A)^r.$$

Using a B-basis for M, the A-linear map $\mathrm{Tr}_{M,B/A} \circ \phi$ is identified with a direct sum of r copies of the A-linear map $B \to \mathrm{Hom}_A(B,A)$ defined by $b \mapsto \mathrm{Tr}_{B/A}(b(\cdot))$. Thus, $\det_A(\mathrm{Tr}_{M,B/A} \circ \phi)$ is equal to the rth power of the "determinant" of this latter A-linear map. By *definition*, the "determinant" (over A) of this map $B \to \mathrm{Hom}_A(B,A)$ is exactly $\mathrm{disc}(B/A)$ (or rather, it is a unit multiple of one of the habitual determinants that generates this ideal, depending on how we choose the A-bases). This completes the proof.