

1. MOTIVATION

A very useful tool in number theory is the ability to construct global extensions with specified local behavior. This problem can arise in several different forms. For example, if  $F$  is a global field and  $v_1, \dots, v_n$  is a finite set of distinct places of  $F$  equipped with finite separable extensions  $K'_i/F_{v_i}$ , and if  $S$  is an auxiliary finite set of non-archimedean places of  $F$ , does there exist a finite separable extension  $F'/F$  unramified at  $S$  such that the completion of  $F'$  at some place  $v'_i$  over  $v_i$  is  $F_{v_i}$ -isomorphic to  $K'_i$ ? Moreover, if all extensions  $K'_i/F_{v_i}$  have degree  $d$  then can we arrange that  $[F' : F]$  has degree  $d$ ? If so, are there infinitely many such  $F'$  (up to  $F$ -isomorphism)?

An alternative problem is the following: if the extensions  $K'_i/F_{v_i}$  are Galois (and so have solvable Galois group, which is obvious when  $v_i$  is archimedean and which we saw in class in the non-archimedean case via the cyclicity of the Galois theory of the finite fields), then can we arrange for  $F'/F$  as above to also be Galois with solvable Galois group? If the extensions  $K'_i/F_{v_i}$  are abelian (resp. cyclic) then can we arrange for  $F'/F$  to have abelian (resp. cyclic) Galois group?

A typical example of much importance in practice is this: construct a totally real solvable extension  $L/\mathbf{Q}$  that is unramified over a specified finite set of primes  $S$  of  $\mathbf{Q}$  and induces a specified extension of  $\mathbf{Q}_p$  for  $p$  in another disjoint finite set of primes. Here we are prescribing the local behavior of  $L$  at a finite set of primes  $p$  and also at the infinite place while insisting that  $L$  also be unramified at the auxiliary finite set  $S$ . In this handout we provide an affirmative answer to all such construction problems, though we must note at the outset that whereas the answer to the first problem (construction of  $F'/F$  with prescribed local behavior but no Galois conditions) will only require the weak approximation theorem (as in §9 of the handout on absolute values), the solution of the problems involving Galois groups will reduce to the abelian case whose solution requires class field theory (and so we will give suitable references in the book of Artin–Tate).

To indicate the subtle nature of such construction problems, consider the cyclic case of the preceding questions: if  $K'_i/F_{v_i}$  is cyclic with degree  $n_i$  for all  $i$ , can we find a cyclic extension  $F'/F$  inducing  $K'_i/F_{v_i}$  for each  $i$ ? If such a construction can be done, then the cyclic group  $\text{Gal}(K'_i/F_{v_i})$  is the decomposition group at  $v_i$  in the cyclic group  $\text{Gal}(F'/F)$ , and so  $[F' : F]$  must be divisible by the least common multiple  $n$  of the  $n_i$ 's. Can one always construct such a cyclic extension  $F'/F$  with this minimal possible degree? It turns out that this is a subtle problem and in very special circumstances it has a negative answer, and then the best that can be done is  $[F' : F] = 2n$ . These special circumstances are exhaustively studied in Chapter X of Artin–Tate, and they only arise when  $F$  is a number field and some of the  $v_i$ 's are non-archimedean with residue characteristic 2. We simply give an explicit counterexample here:  $F = \mathbf{Q}$ ,  $\{v_i\} = \{v_1\} = \{2\}$ , and  $K'_1/\mathbf{Q}_2$  the unramified extension of degree 8. There does *not* exist a cyclic extension  $F'/\mathbf{Q}$  of degree 8 in which 2 is inert (which is to say that  $F'$  induces the degree-8 unramified extension of  $\mathbf{Q}_2$ ). The proof of such non-existence can be given using class field theory, via the product formula for the norm residue symbol in cyclic extensions (see the discussion following Theorem 1 in §1, Chapter X in Artin–Tate). It can also be given in an elementary way as a consequence of the determination of  $(2|p)$  for odd primes  $p$ ; we leave this latter approach to non-existence as an exercise.

2. APPROXIMATION WITH ONE LOCAL FIELD

Before we attack the problem of finding global extensions that induce given local extensions, we address an issue that we should have considered some time ago: is every local field  $K$  realized as a completion of a global field  $F$ ? The archimedean case is obvious, and in the non-archimedean cases we treat characteristic 0 and positive characteristic separately, as follows. If  $K$  has positive characteristic then  $K \simeq k((t))$  for a finite field  $k$ , and so we may take  $F = k(t)$  and  $v$  to be the  $t$ -adic place of  $F$ . If  $K$  has characteristic 0 and residue characteristic  $p$  then  $K$  is a finite (separable) extension of  $\mathbf{Q}_p$ , and since  $\mathbf{Q}_p$  is the  $p$ -adic completion of  $\mathbf{Q}$  we may simply invoke the following theorem with  $F = \mathbf{Q}$  and  $v$  the  $p$ -adic place. (Note that for this application to  $p$ -adic fields we only need the separable aspect of the theorem below, and this aspect is handled near the end of the proof by a simple self-contained application of Krasner's lemma.)

**Theorem 2.1.** *Let  $F$  be a global field and let  $v$  be a non-trivial non-archimedean place on  $F$ . Let  $K'$  be a finite extension of the local field  $K = F_v$ , say with degree  $d$ . There exists a finite extension  $F'/F$  with degree  $d$  and a place  $v'$  on  $F'$  over  $v$  such that  $F'_{v'}$  is isomorphic to  $K'$  over the identification  $F_v = K$ . If  $K'/K$  is separable then  $F'/F$  must be separable.*

*If  $K'/K$  is Galois, then there exists a finite Galois extension  $F'/F$  with a place  $v'$  over  $v$  and an intermediate field  $F_0$  whose completion under  $v'$  is identified with  $F_v = K$  such that  $\text{Gal}(F'/F_0)$  is identified with  $\text{Gal}(K'/K)$ .*

Note in particular that if  $K'/K$  is abelian then we can realize it as a completion of an abelian extension of global fields with the same Galois group. However, this comes at a serious cost: we have to replace the initial  $F$  with some unknown extension  $F_0$ . In the next section we will give a better solution to a generalization of the refined problem with Galois groups, avoiding the interference of the auxiliary  $F_0$ , by using the full power of class field theory.

Historically, the ability to realize an abelian extension of local fields as the completion of an abelian extension of global fields (with the same Galois group, but with weak control on the global base field) was important in the first approach to local class field theory via global class field theory. This is the approach followed in Lang's *Algebraic Number Theory*, but such an approach forces one to confront questions concerning dependence of the resulting theory on the choice of "global model" for a given local abelian extension. The approach to local class field theory via Galois cohomology (as in Serre's *Local fields*) is intrinsic to the local case and so avoids such hassles.

The reader will check that in the separable case, the proof below has nothing to do with global fields and so the result in the separable case is really a theorem about fraction fields of arbitrary discrete valuation rings.

*Proof.* Suppose we have such an  $F'$ . Let us show first that  $F'/F$  must be separable with  $v'$  the unique place on  $F'$  over  $v$  if  $K'/K$  is separable, in which case when  $F'/F$  is Galois then  $\text{Gal}(F'/F) = D(v'|v)$  obviously maps isomorphically to  $\text{Gal}(K'/K)$ . Since  $F'_{v'} = F'F_v = F'K$ , the natural map  $F' \otimes_F F_v \rightarrow F'_{v'}$  over  $F_v$  is surjective. Consideration of  $F_v$ -dimensions shows that this must be an isomorphism, and hence if  $F'_{v'}/F_v$  is separable then the discriminant of the  $F_v$ -algebra  $F' \otimes_F F_v$  is nonzero. Since the formation of the discriminant is compatible with extension of the base field, it follows that the discriminant of the  $F$ -algebra  $F'$  must be nonzero, and from the handout on étale algebras we know that this latter non-vanishing is equivalent to separability of the field extension  $F'/F$ . Since  $F'/F$  is separable,  $F' \otimes_F F_v$  is  $F_v$ -isomorphic to the product of the completions of  $F'$  at the places over  $v$  on  $F'$ , so since this tensor product is identified with  $F'_{v'}$  we conclude that  $v'$  is indeed the unique place on  $F'$  over  $v$ .

Now we turn to the existence problem. By working in towers, we may separately treat the cases when  $K'/K$  is separable and (in characteristic  $p > 0$ ) purely inseparable. We first dispose of the purely inseparable case, say with characteristic  $p > 0$ . We may assume  $K'/K$  has degree  $p$ , so  $K' \subseteq K^{1/p}$ . Since  $K \simeq k((t))$  with  $k$  perfect (even finite) of characteristic  $p$ , it is clear that  $K^{1/p} = k((t^{1/p})) = K(t^{1/p})$  has degree  $p$  over  $K$ , so  $K' \simeq K^{1/p}$  over  $K$ . In other words,  $K'/K$  is unique up to isomorphism. Let  $\kappa$  be the constant field of  $F$  and use Riemann–Roch to exhibit  $F$  as a finite separable extension of the field  $F_0 = \kappa(t)$  such that  $v$  lies over the  $t$ -adic place of  $F_0$ . Note that  $t$  is not a  $p$ th power in  $F$  (as it is not a  $p$ th power in the subfield  $F_0$  and  $F/F_0$  is separable). Hence,  $F' = F(t^{1/p})$  is a purely inseparable extension of degree  $p$ . There is a unique place  $v'$  on  $F'$  over  $v$  (why?) and  $F'_{v'}$  contains a  $p$ th root of  $t$ . If we can show that  $t$  is not a  $p$ th power in  $F_v$  then it follows that the extension  $F'_{v'} = F_v(t^{1/p})$  is purely inseparable of degree  $p$  and so realizes  $K'/K$ . Since  $v$  lies over the  $t$ -adic place on the subextension  $F_0 = \kappa(t)$  over which  $F$  is finite separable,  $F_v$  is finite separable over the  $t$ -adic completion  $\kappa((t))$  of  $F_0$ . Hence,  $t$  is a  $p$ th power in  $F_v$  if and only if it is a  $p$ th power in  $\kappa((t))$ , and so the obvious falsehood of this latter possibility shows that indeed  $t$  is not a  $p$ th power in  $F_v$ , as desired. This takes care of inseparability aspects. (Using stronger methods in commutative algebra, it can be proved that for any global function field  $F$  with characteristic  $p$  and any non-trivial place  $v$  on  $F$ , the non-algebraic extension of fields  $F_v/F$  is always separable in the sense of field theory, and so in particular an element of  $F$  that is not a  $p$ th power cannot become a  $p$ th power in  $F_v$ . This is best understood via Grothendieck's theory of excellent rings. The point of mentioning it here is to clarify that the preceding *ad*

*hoc* trick with Riemann–Roch is not really necessary if we grant ourselves better foundations in commutative algebra.)

It remains to study the more interesting (and more important) case when  $K'/F_v$  is a finite separable extension. By the primitive element theorem, we may write  $K' \simeq F_v(\alpha)$  where  $\alpha$  is a root of an irreducible separable monic polynomial  $h \in F_v[T]$  with degree  $d$ . By Krasner's lemma, if  $h_1 \in F_v[T]$  is a monic polynomial of degree  $d$  whose coefficients are sufficiently close to those of  $h$ , then  $h_1$  is separable and irreducible with the field  $F_v[T]/(h_1)$  isomorphic to  $F_v[T]/(h) \simeq F_v(\alpha)$  over  $F_v$ . In other words, slightly moving  $h$  as a monic polynomial of degree  $d$  does not affect its usefulness as a way to describe  $K'$  over  $F_v$ . Hence, by denseness of  $F$  in  $F_v$ , we may assume  $h \in F[T]$ . Irreducibility and separability of  $h$  over  $F_v$  forces the same over the subfield  $F$ , and thus  $F' = F[T]/(h)$  is a finite separable extension of  $F$  with degree  $\deg h = d$ . Moreover, we have an  $F_v$ -algebra isomorphism  $F_v \otimes_F F' \simeq F_v[T]/(h) \simeq K'$ , yet since  $F'/F$  is a finite separable extension we have  $F_v \otimes_F F' \simeq \prod_{v'|v} F_{v'}$ . Hence, it follows that  $v$  admits a unique extension to a place  $v'$  on  $F'$ , and that  $F_{v'}$  is  $F_v$ -isomorphic to  $K'$ .

Finally, assume that  $K'/K$  is Galois. The finite separable extension  $F'/F$  as above is generally not Galois, so let  $F''/F'$  be a Galois closure of  $F'$  over  $F$ . Since  $F''/F$  is a splitting field for  $h$ , upon choosing a place  $v''$  on  $F''$  over  $v'$  we see that  $F_{v''} = F''F_v$  is a splitting field over  $F_v = K$  for  $h$ . However, the subfield  $K' = F_{v'}$  is such a splitting field since  $K'/K$  is Galois and is generated by a root of  $h$ , so  $F_{v''} = F_{v'}$ . Taking  $F_0$  to be the decomposition field for  $v''$  over  $F$ , and  $v_0$  to be the place on  $F_0$  under  $v''$ , the general theory of the decomposition field shows that  $v''$  is the unique place on  $F''$  over  $v_0$  and  $D(v''|v) = D(v''|v_0)$ , with  $F_{v''} = (F_0)_{v_0} \otimes_{F_0} F''$ , and the resulting natural map

$$\text{Gal}(K'/K) = \text{Gal}(F_{v''}/F_v) = \text{Gal}(F_{v''}/(F_0)_{v_0}) \rightarrow \text{Gal}(F''/F_0) = D(v''|v_0) = D(v''|v)$$

an isomorphism. Hence, by completing  $F''/F_0$  at  $v''$  and  $v_0$  we recover  $K'/K$  without changing the Galois group.  $\blacksquare$

### 3. APPROXIMATION WITH SEVERAL LOCAL FIELDS

Having shown that any local field may be obtained by completing a global field at a suitable place, we can now turn to our main questions considered at the outset: can we simultaneously approximate extensions of finitely many completions of a fixed global field  $F$  with completions of a single finite extension of  $F$ ? In other words, can we construct global extensions with prescribed local behavior? And how about refinements concerning the structure of local and global Galois groups? It is this simultaneous local approximation that will make use of weak approximation, as well as class field theory for the Galois aspects.

Let  $v_1, \dots, v_n$  be distinct places of  $F$  and let  $K'_i/F_{v_i}$  be a finite extension. Can we find a finite extension  $F'/F$  and a place  $v'_i$  on  $F'$  over  $v_i$  such that  $F'_{v'_i} \simeq K'_i$  over  $F_{v_i}$  for every  $i$ , with  $F'/F$  even separable (resp. Galois) when all  $K'_i/F_{v_i}$  are separable (resp. Galois)? This is a refinement of Theorem 2.1 in several respects: we permit several valuations at once, we allow archimedean places, and we insist on a *fixed* global base field  $F$ . If  $K'_i/F_{v_i}$  is separable for some  $i$  but inseparable for others then  $F'/F$  cannot be separable, so in such cases the simultaneous approximation is impossible: it can be proved that non-trivial inseparability in an extension of global function fields *cannot* be lost under passage to any completions. Likewise, if the inseparability degrees of  $K'_i/F_{v_i}$  are not all the same, the construction cannot be done. Hence, it is only reasonable to consider the cases when all  $K'_i/F_{v_i}$  are separable or all  $K'_i/F_{v_i}$  are inseparable with the same nontrivial degree of inseparability. Since local fields  $K$  of characteristic  $p > 0$  admit exactly one purely inseparable extension of degree  $p^n$  for each  $n > 0$  (namely,  $K^{1/p^n}$ ), simultaneous approximation of inseparable local extensions with a common inseparability degree (which never comes up in practice, as far as I am aware) is easily reduced to the separable case. It is the separable case that is of fundamental importance, and so only this case will be considered in what follows.

**Theorem 3.1.** *Let  $F$  be a global field and let  $v_1, \dots, v_n$  be inequivalent places on  $F$ . Let  $K_i = F_{v_i}$ , and let  $K'_i/K_i$  be finite separable extensions. There exists a finite separable extension  $F'/F$  and places  $v'_i$  on  $F'$  over  $v_i$  such that  $F'_{v'_i}$  is  $K_i$ -isomorphic to  $K'_i$  for every  $i$ , and if  $[K'_i : K_i] = d$  for all  $i$  then we can arrange*

that  $[F' : F] = d$ . If  $S$  is a finite set of non-archimedean places of  $F$  distinct from the  $v_i$ 's then  $F'/F$  may be arranged to also be unramified over all places in  $S$ .

If moreover all  $K'_i/K_i$  are Galois then we can simultaneously arrange that  $F'/F$  is a solvable Galois extension. If in addition all  $K'_i/K_i$  are abelian then  $F'/F$  can be arranged to be abelian, and likewise for cyclic extensions.

Of course, in the Galois case it cannot be arranged that  $[F' : F] = d$  if  $[K'_i : K_i] = d$  for all  $i$  except for possibly if all of the Galois groups  $\text{Gal}(K'_i/K_i)$  are isomorphic (as each such decomposition group at  $v'_i$  must fill up the global group  $\text{Gal}(F'/F)$  for cardinality reasons). This is a very special situation that only seems to be tractable in the important cyclic case, in which case it has a negative answer in special situations; see Theorem 5 in §2, Chapter X in Artin–Tate for the full story concerning the degree aspects in the cyclic case. Our treatment of the Galois assertions in Theorem 3.1 will ultimately reduce to problems in class field theory that are solved in Chapter X of Artin–Tate.

*Proof.* Let  $h_i \in F[T]$  be a monic irreducible separable polynomial such that  $K'_i \simeq K_i[T]/(h_i)$  for each  $i$ ; the existence of such  $h_i$  was shown via Kranser's lemma in the proof of Theorem 2.1 when  $v_i$  is non-archimedean, and it is trivial when  $v_i$  is archimedean (take  $h_i = T$  if  $K'_i = K_i$  and take  $h_i = T^2 + 1$  if  $K_i = \mathbf{R}$  and  $K'_i \simeq \mathbf{C}$ ). Let  $d = \max[K'_i : K_i]$ , so by multiplying  $h_i$  against  $d - \deg h_i$  distinct monic linear  $T - \alpha_{ij}$  for  $\alpha_{ij} \in F$  that are not roots of  $h_i$ , we get monic separable multiples  $H_i \in F[T]$  of  $h_i$  with degree  $d$  such that  $K'_i/K_i$  is generated by a root of  $H_i$ , and in fact  $K'_i/K_i$  is a splitting field of  $H_i$  when  $K'_i/K_i$  is Galois.

Use weak approximation to find a monic polynomial  $h \in F[T]$  that is  $v_i$ -close to  $H_i$  for every  $i$ , so in particular  $h$  is separable. Moreover, for some  $i$  we have  $H_i = h_i$ , and hence  $H_i$  is irreducible in  $F_{v_i}[T]$  for such an  $i$ . Thus, by using sufficiently close approximation at this place we ensure that  $h$  is irreducible over  $F_{v_i}$  and so is irreducible over  $F$ . (Strictly speaking, this irreducibility step supposes that the distinguished place  $v_i$  is non-archimedean, but the archimedean analogue is trivial: the only non-linear monic irreducible polynomials in the archimedean case occur for monic quadratic polynomials in  $\mathbf{R}[T]$  with negative discriminant, and in such cases a small perturbation of the lower-degree coefficients does not affect the sign of the discriminant and so does not affect irreducibility over  $\mathbf{R}$ .) Let  $F' = F[T]/(h)$ , so  $F'/F$  is finite separable with degree  $d$ .

For each  $i$ , we have

$$\prod_{v'_i|v_i} F'_{v'_i} \simeq F_{v_i} \otimes_F F' \simeq F_{v_i}[T]/(h).$$

Since  $h$  and  $H_i$  are separable over  $F_{v_i}$ , by the theorem on continuity of roots (proved in the non-archimedean case in the handout on algebraic closedness of completions of algebraic closures in the non-archimedean case, and provable by elementary local compactness arguments in the archimedean case), it follows that the monic factorization type of  $h$  over  $F_{v_i}$  mirrors that of  $H_i$  in the sense that each monic irreducible factor of  $h$  is close to a unique monic irreducible factor of  $H_i$  with the same degree, with such closeness made as small as we please by taking  $h$  near enough to  $H_i$  in  $F_{v_i}$ . Hence,  $K'_i$  is a factor field of  $F_{v_i}[T]/(h)$  and thus we can find a suitable  $v'_i|v_i$  on  $F'$  for each  $i$  such that  $F'_{v'_i}$  is  $F_{v_i}$ -isomorphic to  $K'_i$ . Moreover, if  $[K'_i : K_i] = d$  for all  $i$  then our construction yields  $[F' : F] = d$ .

Now let  $S$  be a finite set of non-archimedean places away from the  $v_i$ 's, and we ask if  $F'/F$  can be chosen to be unramified at all places in  $S$ . This is very easy: since  $h \in F[T]$  is a monic polynomial of degree  $d$  that is merely constrained by approximation conditions on its lower-degree coefficients when viewed in  $F_{v_i}$  for each  $i$ , weak approximation permits us to arrange for  $h$  to also be simultaneously near any desired monic polynomial of degree  $d$  in  $F_v[T]$  for each  $v \in S$ . There exists a monic irreducible polynomial  $h_v \in \mathcal{O}_{F_v}[T]$  with degree  $d$  and  $\text{disc}(h_v) \in \mathcal{O}_{F_v}^\times$  for every  $v \in S$ , such as a monic lift of a separable irreducible polynomial of degree  $d$  over the finite residue field at each such  $v$ , and so we can arrange that  $h \in F[T]$  is  $v$ -integral and separable irreducible over  $F_v$  with  $\text{disc}(h) \in F^\times$  a  $v$ -adic unit for all  $v \in S$ . It follows that  $F_v \otimes_F F' \simeq F_v[T]/(h)$  is an unramified extension field of  $F_v$  for every  $v \in S$ , so each  $v \in S$  is unramified (and even inert) in  $F'$ .

Assume that  $K'_i/K_i$  is Galois for each  $i$ . We seek to make  $F'/F$  such that  $F'/F$  is also Galois, with Galois group that is even solvable (and moreover abelian when all  $K'_i/K_i$  are abelian, and similarly in the cyclic case). Moreover, we wish to carry out such refined constructions while maintaining unramifiedness

conditions at a fixed finite set of non-archimedean places  $S$  disjoint from the  $v_i$ 's. By the Galois theory of local fields (and the trivial archimedean case), the finite Galois extensions  $K'_i/K_i$  are *solvable*. Suppose we have solved the case when each  $K'_i/K_i$  is abelian. If the solvable extension  $K'_1/K_1$  is not abelian then there exists a non-trivial intermediate field  $L$ , and by induction on field-degrees can find a solvable  $F'/F$  that “works” for the local extensions  $L/K_1$  and  $K'_i/K_i$  for all  $i > 1$ . Now we use induction again, with this  $F'$  as the base field and with the local extensions  $K'_1/L$  and the trivial extensions  $K'_i/K'_i$  for  $i > 1$  (and  $S$  replaced with the set of places over it in  $F'$ ) to get a further solvable extension  $F''/F'$ . The extension  $F''/F$  is finite separable but probably not Galois. However, it is clear (check!) that the Galois closure of  $F''$  over  $F$  is solvable and does the job.

It remains to consider the case when all  $K'_i/K_i$  are abelian. By Theorem 4 in §2, Chapter X of Artin–Tate, there exists a finite abelian extension  $F'/F$  equipped with places  $v'_i$  over  $v_i$  such that  $F'_{v'_i}$  is  $F_{v_i}$ -isomorphic to  $K'_i$  for all  $i$  and such that  $F'$  is unramified over each  $v \in S$ . By Theorem 5 in §2, Chapter X of Artin–Tate, the same holds in the cyclic case. ■

We conclude with two important classes of typical examples, one Galois and one not.

*Example 3.2.* Let  $F$  be global field and let  $v_1, \dots, v_n$  be a set of distinct places on  $F$ . Let  $K'_i/F_{v_i}$  be a finite separable extension of degree  $d > 1$ . We claim that there exist infinitely many non-isomorphic finite separable extensions  $F'/F$  with degree  $d$  such that  $v_i$  is inert in  $F'$  with  $F'_{v_i} \simeq K'_i$  over  $F_{v_i}$  for all  $i$ . By Theorem 3.1, we can use separable Eisenstein polynomials of degree  $d$  at non-archimedean places away from the  $v_i$ 's to make separable extensions  $F'/F$  of degree  $d$  such that each  $v_i$  is inert in  $F'$  with  $F'_{v_i} \simeq K'_i$  over  $F_{v_i}$  for every  $i$  and  $F'/F$  is totally ramified (resp. unramified) at any desired auxiliary finite set of non-archimedean places of  $F$ . Thus, by ramification reasons alone we infer that up to  $F$ -isomorphism there exist infinitely many  $F'/F$  that satisfy the requirements at the  $v_i$ 's because a totally ramified extension of degree  $d > 1$  cannot be unramified.

*Example 3.3.* Let  $F$  be a totally real number field (such as  $F = \mathbf{Q}$ ), and let  $\{v_1, \dots, v_n\}$  be a finite set of non-archimedean places of  $F$ . Let  $K'_i/F_{v_i}$  be a finite Galois extension. We claim that there exists a totally real solvable extension  $F'/F$  inducing  $K'_i/F_{v_i}$  as the local extension at  $v_i$  for each  $i$ , that such  $F'/F$  can be chosen to be unramified (resp. totally ramified) at any auxiliary finite set of places, and that if all  $K'_i/F_{v_i}$  are abelian (resp. cyclic) then  $F'/F$  can be taken to be abelian (resp. cyclic). Of course, as we vary the auxiliary finite set of ramification conditions, we would be getting necessarily non-isomorphic extensions of  $F$ .

Theorem 3.1 provides an affirmative answer to the preceding claim, as we merely insert the archimedean (all real) places of  $F$  into the collection of  $v_i$ 's and at these places we impose the local condition that  $F'/F$  induces the extension  $\mathbf{R}/\mathbf{R}$  at each of the real places of  $F$ . This ensures that  $F'$  is a totally real number field. Of course, we could similarly arrange that  $F'$  is totally complex.