Math 248A. Surjectivity for inertia groups

Let $F$ be complete for a non-trivial discretely-valued place, and let $F''/F'/F$ be a tower of finite extensions with $F''$ and $F'$ Galois over $F$. We give $F'$ and $F''$ their unique places extending the one on $F$, and so inside of $\text{Gal}(F''/F)$ and $\text{Gal}(F'/F)$ there are (by definition) inertia subgroups $I(F''/F)$ and $I(F'/F)$ that are the respective kernels

$$I(F''/F) = \ker(\text{Gal}(F''/F) \rightarrow \text{Aut}(k''/k)), \quad I(F'/F) = \ker(\text{Gal}(F'/F) \rightarrow \text{Aut}(k'/k)).$$

Thus, the natural quotient map $\text{Gal}(F''/F) \rightarrow \text{Gal}(F'/F)$ carries $I(F''/F)$ into $I(F'/F)$. We wish to show that this map of inertia groups is also surjective.

The separable closures of $k$ in $k'$ and $k''$ are the residue fields of the maximal unramified subfields $F'_u$ and $F''_u$ of $F$ in $F'$ and $F''$ respectively, and so the Galois property of these fields over $F$ (why are they Galois?) implies that these respectively separable closures are Galois over $k$. Hence, $k'/k$ and $k''/k$ are normal extensions, and the fixed fields of the inertia subgroups are the respective maximal unramified subextensions (why?). Since $F''_u$ certainly contains $F'_u$, we may replace $F$ with $F'_u$ to reduce to the case when $I(F'/F) = \text{Gal}(F'/F)$ (that is, $F'/F$ has trivial maximal unramified subextension). In this case we need to show that $I(F''/F) = \text{Gal}(F''/F''_u)$ surjects onto $\text{Gal}(F'/F)$. Since $F''_u/F'$ is unramified and Galois yet $F'/F$ is separable with trivial maximal unramified subextension, it follows that $F''_u$ and $F'$ are linearly disjoint over $F$. In other words, the compositum $F''_uF'$ inside of $F''$ is identified with $F''_u \otimes_F F'$ and so the natural injective map

$$\text{Gal}(F''_uF'/F''_u) \hookrightarrow \text{Gal}(F'/F)$$

is an isomorphism. However, the composite map

$$I(F''/F) = \text{Gal}(F''/F''_u) \rightarrow \text{Gal}(F''_uF'/F''_u) \simeq \text{Gal}(F'/F)$$

is exactly the canonical map being considered above, and so its surjectivity is now proved.

In general, if $F_s/F$ is a separable closure and we choose an $F$-embedding of $F'$ into $F_s$, then we use the unique extension of the place on $v$ to a (not discretely-valued nor complete) place on $F_s$ to define the inertia subgroup $I_F$ in $G_F = \text{Gal}(F_s/F)$: it is the subgroup of elements that act trivially on the residue field at this place of $F_s$. It is clear (check!) that $I_F = \varprojlim I(F''/F)$ inside of $G_F = \varprojlim \text{Gal}(F''/F)$ with $F''$ ranging over the finite Galois extensions of $F$ inside of $F_s$. The preceding considerations imply that if $F''/F$ is a fixed finite Galois extension and we fix an $F$-embedding of $F'$ into $F_s$ then $I(F''/F)$ surjects onto $I(F'/F) \subseteq \text{Gal}(F'/F)$ for all $F''/F'$ that are finite Galois over $F$ inside of $F_s$. Hence, since an inverse limit of finite non-empty sets is non-empty (Zorn’s Lemma), the natural map $I_F \rightarrow I(F'/F)$ must be surjective. That is, under the quotient map

$$G_F \twoheadrightarrow \text{Gal}(F'/F)$$

we have $I_F \rightarrow I(F'/F)$.

If we replace “unramified” with “tamely ramified” in everything that went before, and so replace inertia groups $I(F'/F)$ and $I_F$ with wild inertia subgroups $P(F'/F)$ and $P_F$ (and likewise use maximal tamely ramified subextensions to replace the role of maximal unramified subextensions) then everything goes through exactly the same way: $P(F''/F) \rightarrow P(F'/F)$ is surjective for towers $F''/F'/F$ with $F''$ and $F'$ finite Galois over $F$, $P_F = \varprojlim P(F'/F)$ inside of $G_F$, and the canonical map $P_F \rightarrow P(F'/F)$ is surjective for all finite Galois extensions $F'/F$ inside of $F_s$. In this way, the inertia and wild inertia groups have good analogues for $F_s/F$ that are well-behaved with respect to passage to finite level. The same works if we replace $F_s$ with any Galois extension of $F$ (with possibly infinite degree), since all that was used about $F_s$ above was that it is Galois over $F$.

For infinite-degree analogues of the higher ramification groups, the situation is rather more subtle and is bound up with the story of “upper numbering” that is well-explained in Serre’s book *Local fields*.

In number theory, the case of most interest for the preceding considerations is that of non-archimedean local fields. For Galois towers of global fields (or possibly infinite-degree Galois extensions of global fields) we thereby get corresponding surjectivity statements for inertia and wild inertia subgroups at non-archimedean places (upon identifying the associated decomposition groups with Galois groups over a local field).