## Math 248A. Completion of algebraic closure

### 1. Introduction

Let $K$ be a field complete with respect to a non-trivial non-archimedean absolute value $|\cdot|$. It is natural to seek a "smallest" extension of $K$ that is both complete and algebraically closed. To this end, let $\overline{K}$ be an algebraic closure of $K$, so this is endowed with a unique absolute value extending that on $K$. If $K$ is discretely-valued and $\pi$ is a uniformizer of the valuation ring then by Eisenstein's criterion we see that $X^e - \pi \in K[X]$ is an irreducible polynomial with degree $e$ for any positive integer $e$, so $\overline{K}$ has infinite degree over $K$. In particular, $\overline{K}$ with its absolute value is *never* discretely-valued. In general if $K$ is not algebraically closed then $\overline{K}$ *must* be of infinite degree over $K$. Indeed, recall from field theory that if a field $F$ is not algebraically closed but its algebraic closure is an extension of finite degree then $F$ admits an ordering (so $F$ has characteristic 0 and only $\pm 1$ as roots of unity) and $F(\sqrt{-1})$ is an algebraic closure of $F$ (see Lang's *Algebra* for a proof of this pretty result of Artin and Schreier). However, a field $K$ complete with respect to a non-trivial non-archimedean absolute value *cannot* admit an order structure when the residue characteristic is positive (whereas there are examples of order structures on the discretely-valued complete field $\mathbf{R}((t))$ with residue characteristic 0). Indeed, this is obvious if $K$ has positive characteristic, and otherwise $K$ contains some $\mathbf{Q}_p$ and hence it is enough to show that the fields $\mathbf{Q}_p$ do not admit an order structure. For $p > 3$ there are roots of unity in $\mathbf{Q}_p$ other than $\pm 1$, and for $p > 2$ there are many negative integers $n$ that satisfy $n \equiv 1 \bmod p$ and thus admit a square root in $\mathbf{Q}_3$. Similarly, any negative integer $n$ satisfying $n \equiv 1 \bmod 8$ has a square root in $\mathbf{Q}_2$. This shows that indeed $[\overline{K} : K]$ must be infinite if the complete non-archimedean field $K$ is not algebraically closed and its residue field has positive characteristic. The same conclusion holds in the uninteresting case when the residue characteristic is 0, by the following alternative trick. In such cases if we choose a nonzero nonunit $t$ in the valuation ring then $K$ must contain $\mathbf{Q}((t))$ with its $t$-adic valuation, and so $\overline{K}$ contains $L \otimes_{\mathbf{Q}} \mathbf{Q}((t)) = L((t))$ for any finite extension $L/\mathbf{Q}$ inside of $\overline{\mathbf{Q}} \subseteq \overline{K}$. But for any finite Galois extension $L/\mathbf{Q}$ inside of $\overline{\mathbf{Q}}$ we get a finite Galois extension $L((t))/\mathbf{Q}((t))$ inside of $\overline{K}$ with the same Galois group, so $L((t))$ has degree 1 or 2 over the subfield $L((t)) \cap K \subseteq L((t))$ over $\mathbf{Q}((t))$ since $[\overline{K} : K] = 2$. But the equality of Galois groups forces $L((t)) \cap K = F((t))$ for a subfield $F \subseteq L$, and clearly $[L : F]$ is equal to 1 or 2, so $K$ contains such an $F$. In particular, we may choose $L/\mathbf{Q}$ to be a cyclic extension of degree 4 whose unique quadratic subfield is imaginary, so $F$ cannot have an order structure and thus neither can $K$. We therefore get a contradiction if $[\overline{K} : K]$ is finite and $> 1$, even in residue characteristic 0.

Although finite extensions of $K$ are certainly complete with respect to their canonical absolute value (the unique one extending the absolute value on $K$), for infinite-degree extensions of $K$ it seems plausible that completeness (with respect to the canonical absolute value) may break down. Indeed, it is a general fact that $\overline{K}$ is *not* complete if it has infinite degree over $K$. See 3.4.3/1 in the book "Non-archimedean analysis" by Bosch *et al.* for a proof in general, and see Koblitz' introductory book on $p$-adic numbers for a proof of non-completeness in the case $K = \mathbf{Q}_p$. We do not require these facts, but they motivate the following question: is this completion of $\overline{K}$ algebraically closed? If not, then one may worry that iterating the operations of algebraic closure and completion may yield a never-ending tower of extensions. Fortunately, things work out well:

**Theorem 1.1.** *The completion $\mathbf{C}_K$ of $\overline{K}$ is algebraically closed.*

The field $\mathbf{C}_K$ is to be considered as an analogue of the complex numbers relative to $K$, and for $K = \mathbf{Q}_p$ it is usually denoted $\mathbf{C}_p$. Observe that since $\mathrm{Aut}(\overline{K}/K)$ acts on $\overline{K}$ by isometries, this action uniquely extends to an action on $\mathbf{C}_K$ by isometries. The algebraic theory of infinite Galois theory therefore suggests the natural question of computing the fixed field for $\mathrm{Aut}(\overline{K}/K)$ on $\mathbf{C}_K$. Observe that this is not an algebraic problem, since the action on $\mathbf{C}_K$ makes essential use of the topological structure on $\mathbf{C}_K$. It is a beautiful and non-trivial theorem of Tate that if $\mathrm{char}(K) = 0$ and $K$ is discretely-valued with residue field of characteristic $p$ (for example, a local field of characteristic 0) then the subfield of $\mathrm{Gal}(\overline{K}/K)$-invariants in $\mathbf{C}_K$ coincides with $K$. That is, "there are no transcendental invariants" in such cases. This theorem is very important at the beginnings of $p$-adic Hodge theory.

The purpose of this handout is to present a proof of Theorem 1.1. Note that this theorem is proved in Koblitz' book in the special case $K = \mathbf{Q}_p$, but his proof unfortunately is written in a way that makes it seem to use the local compactness of $\mathbf{Q}_p$. The proof we give is a more widely applicable variant on the same method, and we use the same technique to also prove a result on continuity of roots that is of independent interest.

## 2. Proof of Theorem 1.1

Choose $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbf{C}_K[X]$ with $n > 0$. Since $\overline{K}$ is dense in $\mathbf{C}_K$, there exists polynomials

$$f_j = X^n + a_{n-1,j}X^{n-1} + \cdots + a_{0,j} \in \overline{K}[X]$$

with $a_{ij} \to a_i$ in $\mathbf{C}_K$ as $j \to \infty$. If $a_i \neq 0$ then we may arrange that $|a_{ij} - a_i| < \min(|a_i|, 1/j)$ for all $j$, so $|a_{ij}| = |a_i|$ for all $j$. If $a_i = 0$ then we may take $a_{ij} = 0$ for all $j$. Hence, for all $0 \leq i \leq n-1$ we have $|a_{ij}| = |a_i|$ and $|a_{ij} - a_i| < 1/j$ for all $j$. Of course, we have no control over the finite extensions $K(a_{ij}) \subseteq \overline{K}$ as $j$ varies for a fixed $i$.

Since $\overline{K}$ is algebraically closed, we can pick a root $r_j \in \overline{K}$ for $f_j$ for all $j$. The idea is to find a subsequence of the $r_j$'s that is Cauchy, so it has a limit $r$ in the *complete* field $\mathbf{C}_K$, and clearly $f(r) = \lim f_j(r_j) = 0$. This gives a root of $f$ in $\mathbf{C}_K$. Since $f_j(r_j) = 0$ for all $j$, we have

$$|r_j^n| = \left| -\sum_{i=0}^{n-1} a_{ij} r_j^i \right| \leq \max_i |a_{ij}||r_j|^i = \max_i |a_i||r_j|^i$$

because $|a_{ij}| = |a_i|$ for all $j$. Hence, for each $j$ there exists $0 \leq i(j) \leq n-1$ such that $|r_j|^n \leq |a_{i(j)}||r_j|^{i(j)}$, so $|r_j| \leq |a_{i(j)}|^{1/(n-i(j))}$. Thus,

$$|r_j| \leq C \stackrel{\text{def}}{=} \max(|a_0|^{1/n}, |a_1|^{1/(n-1)}, \ldots, |a_{n-1}|)$$

for all $j$. Note that $C$ only depends on the coefficients $a_i$ of $f$.

Since $f$ and $f_j$ are monic with the same degree $n > 0$, we have

$$|f(r_j)| = |f(r_j) - f_j(r_j)| = \left| \sum_{i=0}^{n-1} (a_i - a_{ij}) r_j^i \right| \leq \max_{0 \leq i \leq n-1} |a_i - a_{ij}||r_j|^i \leq \max_{0 \leq i \leq n-1} |a_i - a_{ij}| \cdot \max(1, C^{n-1})$$

because $|r_j|^i \leq C^i \leq C^{n-1}$ for all $i$ if $C \geq 1$ and $|r_j|^i \leq C^i \leq 1$ for all $i$ if $C \leq 1$. Recall that we choose $a_{ij}$ so that $|a_{ij} - a_i| < 1/j$ for all $j$, so we conclude

$$|f(r_j)| \leq \frac{\max(1, C^{n-1})}{j}$$

for all $j$. Hence, $f(r_j) \to 0$ as $j \to \infty$. We shall now use this fact to infer that $\{r_j\}$ has a Cauchy subsequence in $\mathbf{C}_K$, which in turn will complete the proof.

Let $L$ be a finite extension of $\mathbf{C}_K$ in which the monic $f$ splits, say $f(X) = \prod_k (X - \rho_k)$. We (uniquely) extend the absolute value on the (complete) field $\mathbf{C}_K$ to one on $L$, so we may rewrite the condition $f(r_j) \to 0$ as

$$\lim_{j \to \infty} \prod_{k=1}^{n} (r_j - \rho_k) = 0$$

in $L$. In other words, $\prod_{k=1}^n |r_j - \rho_k| \to 0$ in $\mathbf{R}$. Hence, by the pigeonhole principle, since there are only finitely many $k$'s we must have that for some $1 \leq k_0 \leq n$ the sequence $\{|r_j - \rho_{k_0}|\}_j$ has a subsequence converging to 0. Some subsequence of the $r_j$'s must therefore converge to $\rho_{k_0}$ in $L$, so this subsequence is Cauchy in $\mathbf{C}_K$.

## 3. Continuity of roots

Let $f = \sum a_i X^i \in K[X]$ be monic of degree $n > 0$, so the roots of $f$ in $\mathbf{C}_K$ lie in $\overline{K}$. An inspection of the proof of Theorem 1.1 shows that the argument yields the following general result:

**Lemma 3.1.** *Let $\{f_j\}$ be a sequence of monic polynomials $f_j = \sum a_{ij} X^j$ of degree $n$ in $K[X]$ such that $a_{ij} \to a_i$ as $j \to \infty$ for all $0 \le i \le n-1$. Let $r_j \in \overline{K}$ be a root of $f_j$ for each $j$. There exists a subsequence of $\{r_j\}$ that converges to a root of $f$ in $\overline{K}$.*

We may now deduce the following general result that is usually called "continuity of roots" (in terms of their dependence on the coefficients of $f$).

**Theorem 3.2.** *Let $r \in \overline{K}$ be a root of a degree-$n$ monic polynomial $f = \sum a_i X^i \in K[X]$, with $\mathrm{ord}_r(f) = \mu > 0$. Fix $\varepsilon_0 > 0$ such all roots of $f$ in $\overline{K}$ distinct from $r$ have distance at least $\varepsilon_0$ from $r$. (If there are no other roots, we may use any $\varepsilon_0 > 0$.) For all $0 < \varepsilon < \varepsilon_0$ there exists $\delta = \delta_{\varepsilon, f} > 0$ such that if $g = \sum b_i X^i \in K[X]$ is monic with degree $n$ and $|a_i - b_i| < \delta$ for all $i$ then $g$ has exactly $\mu$ roots (with multiplicity) in the open disc $B_\varepsilon(r) = \{x \in \overline{K} \,|\, |x - r| < \varepsilon\}$.*

*Proof.* We argue by contradiction. Fix a choice of $\varepsilon$. If there exists no corresponding $\delta$, then we would get a sequence of monic polynomials $f_j = \sum a_{ij} X^i \in K[X]$ with degree $n$ such that $a_{ij} \to a_i$ as $j \to \infty$ for each $i$ and each $f_j$ does not have exactly $\mu$ roots on $B_\varepsilon(r)$. Pick factorizations $f_j = \prod_{k=1}^n (X - \rho_{jk})$ upon enumerating the $n$ roots (with multiplicity) for each $f_j$ in $\overline{K}$. By Lemma 3.1 applied to $\{\rho_{j1}\}$, we can pass to a subsequence of the $f_j$'s so $\rho_{j1} \to \rho_1$ with $\rho_1$ some root of $f$ in $\overline{K}$. Successively working with $\{\rho_{jk}\}_j$ for $k = 2, \dots, n$ and passing through successive subsequence of subsequences, etc., we may suppose that there exist limits $\rho_{jk} \to \rho_k$ in $\overline{K}$ as $j \to \infty$ for each fixed $1 \le k \le n$.

Each $\rho_k$ must be a root of $f$, but we claim more: every root of $f$ arises in the form $\rho_k$ for exactly as many $k$'s as the multiplicity of the root. Working in the finite-dimensional $\overline{K}$-vector space of polynomials of degree $\le n$ (given the sup-norm with respect to an arbitrary $\overline{K}$-basis, the choice of which does not affect the topology), we have

$$f_j = \prod_{k=1}^n (X - \rho_{jk}) \to \prod_{k=1}^n (X - \rho_k),$$

yet also $f_j \to f$. Hence, $f = \prod_{k=1}^n (X - \rho_k)$ in $\overline{K}[X]$. That is, $\{\rho_k\}$ is indeed the set of roots of $f$ in $\overline{K}$ counted with multiplicites. Hence, $r = \rho_k$ for exactly $\mu$ values of $k$, say for $1 \le k \le \mu$ by relabelling.

By passing to a subsequence we may arrange that for each $1 \le k \le n$, $|\rho_{jk} - \rho_k| < \varepsilon$ for all $j$. In particular, if $1 \le k \le \mu$ we have $|\rho_{jk} - r| < \varepsilon$. Since all roots $r'$ of $f$ distinct from $r$ have distance $\ge \varepsilon_0 > \varepsilon$ from $r$, by the non-archimedean triangle inequality we have $|\rho_{jk} - r'| = |r - r'| \ge \varepsilon_0 > \varepsilon$ for all $1 \le k \le \mu$ and any $j$. However, if $k > \mu$ then $\rho_k$ is such an $r'$, yet $|\rho_{jk} - \rho_k| < \varepsilon$ for all $j$ and all $k$, so for each fixed $j$ we must have $|\rho_{jk} - r| \ge \varepsilon_0 > \varepsilon$ for all $k > \mu$. Thus, for the $j$'s that remain (as we have passed to some subsequence of the original sequence), $\rho_{j1}, \dots, \rho_{j\mu}$ are precisely the roots of $f_j$ (with multiplicity) that are within a distinct $< \varepsilon$ from the root $r$ of $f$. This contradicts the assumption on the $f_j$'s. ■

Here is an important corollary that is widely used.

**Corollary 3.3.** *Let $f \in K[X]$ be a separable monic polynomial with degree $n$. Choose $\varepsilon > 0$ as in Theorem 3.2. For each monic $g \in K[X]$ with degree $n$ and coefficients sufficiently close to those of $f$, $g$ is separable and each root of $g$ in $K_{\mathrm{sep}}$ is within a distance $< \varepsilon$ from a unique root of $f$ in $K_{\mathrm{sep}}$. Moreover, if $f$ is irreducible then $g$ is irreducible.*

*Proof.* We apply Theorem 3.2 with $\mu = 1$ to conclude that if such a $g$ is coefficientwise sufficiently close to $f$ then each of the $n$ roots of $g$ (with multiplicity) is within a distance $< \varepsilon$ from a unique root of $f$. In particular, $g$ has $n$ distinct roots and hence is separable. Thus, all roots under consideration lie in $K_{\mathrm{sep}}$. The uniqueness aspect, together with the fact that $\mathrm{Gal}(K_{\mathrm{sep}}/K)$ acts on $K_{\mathrm{sep}}$ by isometries, implies that the $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-orbit of a root of $g$ has the same size as the $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-orbit of the corresponding nearest root of $f$. Hence, the degree-labelling of the irreducible factorization of $g$ over $K$ "matches" that of the separable $f$, and in particular if $f$ is irreducible then $g$ is irreducible. ■